Competition #15

The Junior Online Math Olympiad

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Short Questions

1. (Guilherme) Let d, e, f be integers such that $d(10e+f)(10f+e) = 2015$. Evaluate def.
2. (Adi) Find the sum all primes p such that $x^2 - 12x + p^k$ has integer roots for some value of k.
3. (ZS) Let $ABCD$ be a square with area 72. Let M, N, O, P be the midpoints of AB, BC, CD, DA respectively. Let W, X, Y, Z be the midpoints of AM, BN, CO, DP respectively. Let E, F, G, H be the intersection of AX and MN, BY and ON, CZ and OP, DW and MP. Find the area of quadrilateral $EFGH$.
4. (ZS) Let ABC be an isosceles triangle with $AB = AC = 13, BC = 10$. Let A_1 be the incircle of ABC. Let A_2 be the circle externally tangent to A_1 and tangent to AB and AC. A_3 is the circle that is tangent to AB, AC and externally tangent to A_2. Define $A_4, A_5, ...$ similarly. If the sum of the circumferences of these circles is $n\pi$, then determine the value of n.
5. (Guilherme) If $1 + 2\sin^2 x = 20\cos^3 x$, evaluate $3 + 4\tan^4 x$.
6. (ZS) Determine the value of $\displaystyle\prod_{i=1}^{2015} [\sin(i^\circ) - \cos(i^\circ)]$.
7. (ZS) Determine the value of $2(\sin^2 1^\circ + \sin^2 2^\circ + ... + \sin^2 89^\circ)$.
8. (Thinula) Consider the polynomial $x^{2015} + 13x - 1 = 0$. Find the sum of the 2015^{th} powers of the roots.
9. (Jo) In $\triangle ABC$, $b\cos C + c\cos b = 10$ and $b\sin C + c\sin B = 18$. Find the area of $\triangle ABC$.
10. (Guilherme) If $\cos\beta + \tan\beta = 1$, find the sum of all possible rational values of $\cos\beta$.

Long Questions

Explain your answer for each question.

1. A quadrilateral $ABCD$ has side lengths $AB = 5, BC = 6, CD = 8, DA = 7$. Prove that there exists a point P in $ABCD$ such that the perpendiculars from P to the sides of $ABCD$ are equal.

   (2 Points)

2. (Navi) Do there exist positive integers x, y such that $\sqrt{5^x + 7^y}$ is an integer?

   (2 points)

3. (ZS) Prove that $3^{2^n} - 1$ can be written as the sum of two squares for all positive integers n.

   (3 points)