Competition #1

The Junior Online Math Olympiad

20th January 2014 - 27th January 2014

Short Questions

1. Three sets, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, have 3, 12, and 13 elements respectively. The set $\mathbb{D}$ is defined as:

$$\mathbb{D} = \{(x, y, z) \mid x \in \mathbb{A}, y \in \mathbb{B}, z \in \mathbb{C}\}$$

How many elements does $\mathbb{D}$ have?

2. How many natural number factors are there of 560?

3. At JOMO's Pizzeria, you can design your own pizza. You can choose from the toppings: Pepperoni, Salami, Onions, Mushrooms, Pineapples, and Ham. You can also choose between a thick and thin crust. A plain Pizza with no toppings costs \$120HKD. Each extra topping you add on costs an extra \$20HKD. If I only have \$200HKD, how many different types of pizza could I order?

Details and Assumptions

You are allowed to order a Pizza with repeating toppings, so you could order a pizza with 3 toppings of all pineapple.

4. A number $x \in \mathbb{R}$ is chosen from the closed interval $[2, 10]$, the probability that $\left\lfloor \frac{x}{2} \right\rfloor$ is even can be expressed as $\frac{a}{b}$, where a and b are positive co-prime integers, find $a + b$

Details and Assumptions

$\lfloor x \rfloor$ is the floor function. $\lfloor x \rfloor$ Is the largest integer not greater than x. For example: $\lfloor 3 \rfloor = 3$, $\lfloor 5.8 \rfloor = 5$, $\lfloor \pi \rfloor = 3$

5. Lorcan the monkey sits on a typewriter and types 6 letters randomly. The probability that he types out "lorcan" on the typewriter can be expressed as $\frac{a}{b}$ where a and b are co-prime positive integers. Find the remainder when $a + b$ is divided by 1000?

   Details and Assumptions

   The typewriter consists of 26 buttons labelled a to z, and no other buttons

6. A slide wraps around a Cylinder of radius $3m$ and height $40\pi m$ exactly 5 times and starts from the top and ends at the bottom. The length of the slide is $x\pi m$. Find x

7. The two graphs $y = x^2 + 4x + 1$ and $y = -3x + 9$ intersect at two points: (a, b) and (c, d). Find the value of $a + b + c + d$

8. Ron the Mouse travels 80 centimeters North and 150 centimeters East. Let d be the distance travelled by Ron the Mouse and s be the magnitude of the displacement of the mouse. Find the value of $d - s$?

9. Three circles have centers: Γ_1, Γ_2 and Γ_3 are tangent to each other, and has Radius 10, 12 and 24 respectively, What is the perimeter of $\triangle \Gamma_1 \Gamma_1 \Gamma_3$?

10. In a city with 6 monuments, you wish to construct one-way roads such that each monument has exactly one road coming in and one road leaving. In how many ways can you do this?

    Details and Assumptions

    Not all 6 monuments need to be inter-connected

    A monument can be connected to itself

Long Questions
Explain your answer for each of the questions

1. Prove that:

$$\frac{\sin 2x}{\cos^2 x} = 2 \tan x$$

2. For all positive reals a and b, Let

$$\sqrt{\frac{(a + b)^2 + (a - b)^2}{2}} = c$$

   Prove that a, b, and c are side lengths of a right angled triangle

3. Let $x, y, z \geq -1$ be real numbers satisfying $xy + yz + zx + xyz = 2$. Find the minimum value of $(1 + x)(1 + y)(1 + z)$ and prove that it is the minimum.