Competition #2

The Junior Online Math Olympiad

17th February 2014 - 24th February 2014

Short Questions

1. How many positive integers n are there such that $6^n - 1$ is a prime number?

2. Find the remainder when:

$$\left(\prod_{k=0}^{30}(16+13k)\right)+\left(\sum_{i=0}^{10}(140-13i)\right)$$

   Is divided by 13

3. Alfred and Boris have a 70 km race. Alfred runs at a constant velocity of 7 km h^{-1}, While Boris runs at a constant velocity of 10 km h^{-1}. However, Boris trips during the contest, after which he goes at a constant velocity of 5 km h^{-1} as his knee hurts. We do not know when Boris tripped. The probability that Alfred won the race can be expressed as $\frac{a}{b}$ where a and b are positive co-prime integers. Find the value of $a+b$

4. Let x, y, and z be positive reals satisfying the following:

$$\begin{aligned} z &= 2x+4y+5 \\ -x^2+xy+y^2 &= 2z^2+8 \\ 5x^2+3xy &= -8-z^2 \end{aligned}$$

   If $y = -\frac{a}{b}$ for positive co-prime integers a and b, find $a+b$

5. A squircle is a mathematical shape with properties between those of a square and those of a circle. A squircle centered at the origin is defined by the equation: $x^4+y^4=r^4$

   What is the sum of the squares of the x-coordinates of the intersections of the graphs: $x^4+y^4=8$ and $y=\sqrt[4]{-6x^2+16}$

6. Find the sum of the solutions of x for:

$$5^{2x} + 5^5 = 5^{x+3} + 5^{x+2}$$

7. Cruella De Vil has 15 dalmatians captive. She wishes to cage them in groups of 3. If each dalmatian is distinct, and the number of ways Cruella can arrange the 15 dalmatians in each of the cages is n, find the sum of the digits of n.

   Details and Assumptions

   Order inside the cages does not matter, but the cages are numbered, so the order of cages does matter.

8. In $\triangle ABC$, $AB = 10$ and $AC = 12$, D is a point on line BC such that line AD bisects $\angle BAC$, The ratio $\frac{BD}{CD}$ can be expressed as $\frac{p}{q}$, where p and q are positive co-prime integers, find the value of $p + q$

9. A circular dart board has a radius of 9cm, the board is split into 3 regions. The first region is enclosed in a circle centered in the center of the dart board with a radius of 2cm. The second region is in between the circle that defines the first region and another circle also centered at the center of the dart board and with a radius of $x$cm. The third region is the rest of the dart board that is not occupied by the first or second region.

   Monica the Monkey throws a dart at the dartboard. The Probability that the dart hits the second Region is $\frac{5}{9}$. Find the value of x.

   Details and Assumptions

   Assume that Monica the Monkey's dart hits the board everytime and that the dart has equal probability to hit any point.

10. Let $|x|$ be the answer to this question. Find the value of $\frac{1}{x}$.

Long Questions

Explain your answer for each of the questions

1. Show that all numbers whose digits are a permutation of 1234567890 are not prime.

   (2 Points)

2. Ronald the Mouse stands on the bottom leftmost corner on a $n \times n$ units^2 grid. He can only move 1 unit up or 1 unit right at a time, prove that the number of paths Ronald the Mouse can take to get to the upper right corner will always be in the form $\binom{2n}{n}$ for any $n \times n$ grid

   (4 points)

3. Prove that $\frac{1}{2}y^2 + 5x^2 + \frac{5}{2}y + 3x - 2xy + \frac{17}{2} \geq 0$ for all $x, y \in \mathbb{R}$.

   (3 points)