Competition #4

The Junior Online Math Olympiad

28th April 2014 - 5th May 2014

Short Questions

1. Let S denote the set of numbers where each number is a permutation of the digits $1, 2, 3, 4, 5, 6, 7, 8$. A number x is chosen randomly from the set S. The probability that x is divisible by 36 is $\frac{m}{n}$ where m and n are positive coprime integers. Find the value of $m + n$.

2. Ron wants to create a song that consists of the following phrases:

   - "Aashyman Aashyman"
   - "Jacku Jacku"
   - "Carly Carly"
   - "Csajah Csajah"
   - "Curry Curry"
   - "Chikun Chikun"
   - "Chhhhhikuuuuuuuuuun"
   - "Nowy Nowy"
   - "Ronnam Ronnam Nowy Nowy Ronnam Wisda Taichi!"
   - "Eh Eh"

   He wants the song to have 4 *verses*, and each verse to have 3 *phrases*. Each phrase has to appear at least once.

   Moreover, he wants the phrases "Ronnam Ronnam Nowy Nowy Ronnam Wisda Taichi!", "Eh Eh"and "Chhhhhikuuuuuuuuuun" to always be the last phrase of a verse.

   Let the Number of songs Ron can make be a, find the remainder when the product of the non-zero digits of a is divided by 1000.

3. When a mp3 is on shuffle, after a song ends the mp3 player will choose a song at random and play it, the mp3 will not play the same song twice in a row.

   Yan Yau has 10 distinct songs on his mp3. One day he goes on his mp3 and listens to 5 song on shuffle. The probability that all the songs he listened to are distinct can be expressed as $\frac{m}{n}$ where m and n are positive coprime integers. Find the value of $\left\lfloor \frac{m+n}{1000} \right\rfloor$.

   Details and Assumptions

   $\lfloor x \rfloor$ is the floor function. $\lfloor x \rfloor$ Is the largest integer not greater than x. For example: $\lfloor 3 \rfloor = 3$, $\lfloor 5.8 \rfloor = 5$, $\lfloor \pi \rfloor = 3$

4. If a permutation of all 26 letters of the english alphabet is chosen at random, what is the probability that the permutation contain the letters MATHS in the same order (A doesn't come before M , T doesn't come before A , H doesn't come before T , S doesn't come before H). The probability can be expressed as $\frac{m}{n}$ where m and n are positive coprime integers. Find the value of $m + n$.

   Details and Assumptions

   For example: "MATHSBCDEFGIJKLNOPQRUVWXYZ" is a valid sequence and so is "MPAQTRHSBCDEFGIJKLNOUVWXYZ", as order of letters of MATHS is maintained

5. Tom is the Burger Master and he just gave me a secret recipe of how to prepare the most delicious burger in the world. First, take a bun as the base. Then, add a meatloaf on it. Lastly, put another bun on top of it and we are done. Today, Tom gets an order for 3 burgers. Find the number of distinct ways are there for Tom to produce these 3 burgers divided by 10

   Details and Assumptions

   Tom doesn't need to finish the first burger to continue to the next burger, he also doesn't need to finish all the base to continue to the meatloaf.

   For example: He can first put on the base of the first burger, then put on the base for the second burger, then put on the meatloaf for the first burger, then put on the base for the third burger.. etc...

6. Aditya, Ben and Cody are playing basketball. Aditya starts the ball. How many ways are there for them to pass the ball such that Aditya receives the ball at the 8th pass?

   Details and Assumptions

   No one can pass the ball to themselves

7. If you draw two cards from a standard 52-card deck without replacement, then the probability that they are different colors is equal to $\frac{A}{B}$, where A and B are positive coprime integers. Find $A + B$

8. There are 38 married couples (a couple is a man and woman) in a party and for a game, they need to be divided into groups of 19 couples each, (i.e. a husband and a wife won't be in different groups). There are 3 men who are best friends and there are 4 women (other than the wives of the 3 men) who are best friends. The number of different groups can be made such that all the who are best friends remain in same group is n, find the digit sum of n

9. 20 students participate in a school-made olympiad: everyone has to write a question on a numbered piece of paper (from 1 to 20) and put it in a bag. The students proceed to get their to-be-solved questions by rolling a 20-sided fair dice. Let the probability that no student gets their own question be n. Find the value of $\lfloor 1000n \rfloor$

   Details and Assumptions

   - One question can be given to many people by the die
   - $\lfloor x \rfloor$ is the floor function. $\lfloor x \rfloor$ Is the largest integer not greater than x. For example: $\lfloor 3 \rfloor = 3$, $\lfloor 5.8 \rfloor = 5$, $\lfloor \pi \rfloor = 3$

10. Yan Yau wants to paint eggs for Easter. He has 20 eggs and three colors, red green and blue. He wants the number of red eggs to be a multiple of three, green eggs to be a multiple of 4, and blue eggs to be a multiple of 5. If the number of eggs total that he paints is a multiple of 4, how many way can he paint eggs?

    Details and Assumptions

    Note that he can choose not to paint some eggs.

Long Questions

Explain your answer for each of the questions

1. Prove that given any 37 positive integers it is possible to choose 7 whose sum is divisible by 7.

   (3 points)

2. Inside a cube of side 5 units, there are 260 points drawn. Prove that there exists a sphere of unit radius within this cube such that 4 of these points are within the sphere.

   Details and Assumptions

   Unit radius means radius of 1 unit.

   (3 points)

3. Sam has a sequence of n consecutive positive integers $a_1, a_2, a_3 \cdots a_n$ where n is an odd number and a_1 is an odd number. Adi arranges all of Sam's

numbers in some permutation. Prove that given any permutation of the sequence $(p_1, p_2, p_3 \cdots p_n)$. The expression:

$$\left(\prod_{k=1}^{\frac{n-1}{2}} \left(p_{2k} + 6^k\right)\right) \left(\prod_{k=1}^{\frac{n+1}{2}} \left(p_{2k-1} + 17^k\right)\right)$$

will always be divisible by 2.

Details and Assumptions

Basically, in the expression p_a is added to $6^{\frac{a}{2}}$ if a is even and $17^{\frac{a+1}{2}}$ if a is odd. Then all the brackets are multiplied together.

(2 points)