Competition #5

The Junior Online Math Olympiad

26th May 2014 - 3rd June 2014

Short Questions

1. Find the **last 3 digits** of the greatest possible integer value of m such that $m^5 + 5^5$ is divisible by $m - 5$

2. x, y, and z are non-negative integers where $xy + 3x + 2y = 2$ and $yz + 4y + 3z = 52$. Find the value of $x + y + z + xy + yz + xz + xyz$

3. Evaluate, to the nearest integer, the sum of the perimeter and the area of the figure when $x^{2500000} + y^{2500000} \leq 2^{2500000}$ is drawn on a Cartesian plane.

4. Sam has 3 numbers $a, b, c > 0$. Sam has stated that the numbers follow the condition:

$$ab + bc + ac = abc$$

   Adi has been assigned the task of finding the smallest possible value of $a + b + c$. Can you help him by finding this value?

5. Given that p and $p^2 + 8$ are both primes , what is the sum of all possible values of p ?

6. Given that $7 \nmid xy$ and $7 \mid 2x + 3y$, find the **smallest positive** value of k such that $7 \mid kx + 282y$

7. Evaluate the last 3 digits of 201^{402}

8. Let $f : \mathbb{N} \to \mathbb{N}$ be defined as:

$$f(n) = (((n!)!)!)!$$

   Find the remainder when $f(1) + f(2) + f(3) + \ + f(2014)$ is divided by 2014

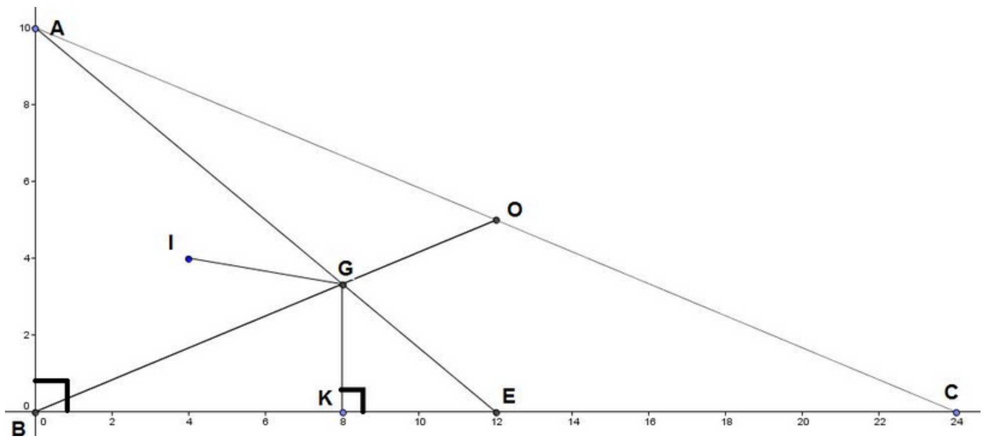
9. One day, 7 friends decided to find a rare Mathematics book (Imagine the ancient Fermats notebook). To do so, they sourced out 5 oldest bookshops in the world. However, due to time constraint (they have to get the book before someone else does), they decided to split themselves up such that there is at least 1 person who visits each bookshop. Find the number of ways that they can arrange themselves divided by 100.

   Details and Assumptions

   The 7 friends are considered distinct people. Same applies for the 5 bookshops

10. In the Diagram, $AB = 10, BC = 24$. I is the Incenter of $\triangle ABC$, G is the Centroid of $\triangle ABC$. O and E are the midpoints of AC and BC respectively.

    The product of $OG, BG, BI, IE, OE, GE, GK, KE$, and IK can be expressed as $\frac{a\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Z}^+$, $gcd(a, c) = 1$, and b is not divisible by the square of a integer larger than 1. Find the last 3 digits of $a + b + c$



Long Questions

Explain your answer for each of the questions

1. Triangle $\triangle ABC$ has right angle at C such that $AC^2 + BC^2 = AB^2$. Squares $ACDE$ and $BCFG$ are constructed on the exterior of of ABC. Show that $DF = AB$

   (3 Points)

2. Prove that there is a real value of x real such that $\sin x + \cos x + \tan x = 0$

   [no drawn/approximated answers will be accepted]

   (2 points)

3. Prove that

$$\sin^7\theta+\cos^7\theta = (\sin\theta+\cos\theta)\left(\sin^4\theta+\cos^4\theta-\frac{(1+\sin\theta+\cos\theta)(\sin\theta+\cos\theta-1)(1-\sin^2\theta\cos^2\theta)}{2}\right)$$

Note: For you proof let $\sin\theta = s$ and $\cos\theta = c$

(3 points)