Competition #1 Suggested Solutions

The Junior Online Math Olympiad

Answer Key:

1. 468
2. 20
3. 420
4. 3
5. 777
6. 50
7. 32
8. 60
9. 92
10. 720

Short Questions

1. Set $\mathbb{D}$ is defined as the set of triplets (x, y, z) where x belongs to $\mathbb{A}$, y belongs to $\mathbb{B}$, and z belongs to $\mathbb{C}$, There are 3 elements in $\mathbb{A}$, 12 elements in $\mathbb{B}$, 13 elements in $\mathbb{C}$. There are 3 possibilities for x, 12 possibilities for y, and 13 possibilities for z. So therefore there are:

$$3 \times 12 \times 13 = \boxed{468} \text{ Elements in } \mathbb{D}$$

2. The Prime Factorisation of 560 is: $560 = 2^4 \times 5 \times 7$

   Each factor of 560 can either have no factors of 2, 1 factor of 2, 2 factors of 2, 3 factors of 2, or 4 factors of 2, so there are 5 possibilities. There could either be 1 factor of 5 or no factors of 5 so there are 2 possibilities. There is either 1 factor of 7 or no factors of 7, so there are 2 possibilities. So there are $5 \times 2 \times 2 = \boxed{20}$ factors of 560

3. The Pizza I order can have a maximum of 4 toppings.

   If there are no toppings, I can order 2 different pizzas, one with thick crust and one with thin crust.

   With 1 topping, There are $C_1^6 = 6$ different toppings you can choose, and you can choose between thick and thin crust. So there are 12 different pizzas you can order with 1 topping.

   With 2 toppings, you can have 2 different toppings or two identical toppings, which has $C_2^6 = 15$ and $C_1^6 = 6$ different combinations respectively. You can choose between thick and thin crust. So there are $(15+6)\times 2 = 42$ different pizzas you can order with 2 toppings.

   With 3 toppings, you can have 3 different toppings, two identical toppings and 1 different, and 3 identical toppings, which has $C_3^6 = 20$, $C_1^6 C_1^5 = 30$, and $C_1^6 = 6$ different combinations respectively. You can choose between thick and thin crust. So there are $(20 + 30 + 6) \times 2 = 112$ different pizzas you can order with 3 toppings.

   With 4 toppings, you can have 4 different toppings, 2 identical toppings and 2 different, 2 identical toppings and another 2 identical toppings, 3 identical toppings and 1 different, and 4 identical toppings, which has $C_4^6 = 15$, $C_1^6 \frac{C_1^5 C_1^4}{2!} = 60$, $\frac{C_1^6 C_1^5}{2!} = 15$, $C_1^6 C_1^5 = 30$ and $C_1^6 = 6$ different combinations respectively. You can choose between thick and thin crust. So there are $(15 + 60 + 15 + 30 + 6) \times 2 = 252$ different pizzas you can order with 4 toppings.

   In total there are $2 + 12 + 42 + 112 + 252 = \boxed{420}$ Different Pizzas I can order with \$200 HKD

4. The length of the interval $[2, 10]$ is 8 units. If $4 \le x < 6$ or $8 \le x < 10$ then $\left\lfloor \frac{x}{2} \right\rfloor$ is an even integer. The sum of the lengths of the interval $[4, 6)$ and $[8, 10)$ is 4.

   By Geometric Probability, the probability that $\left\lfloor \frac{x}{2} \right\rfloor$ is even is $\frac{4}{8} = \frac{1}{2}$. Therefore $a = 1$ and $b = 2$, and $a + b = \boxed{3}$

5. There are 26 possibilities for each letter, and there are 6 letters so there are $26^6 = 308915776$ different combinations that Lorcan the Monkey could have typed. Out of these combinations 1 of them is "lorcan" therefore the probability that he types out "lorcan" on the typewriter is $\frac{1}{308915776}$ So $a = 1$ and $b = 308915776$. Therefore $a + b = 308915777$. The remainder when $a + b$ is divided by 1000 is 777, so the answer is $\boxed{777}$

6. Since we know the Radius is 3m the circumference of the base is 6πm. The slide wraps around the cylinder 5 times so the length of the base is 30πm. The height is 40πm. So by the Pythagorean Theorem we get that the length of the slide is $\boxed{50}\pi$m

7. By solving simultaneous equations, we get:

$$x^2 + 4x + 1 = -3x + 9$$

Now solve for x:

$$x^2 + 7x - 8 = 0$$
$$(x+8)(x-1) = 0$$
$$x = -8 \text{ OR } x = 1$$

Substituting x into the original functions, the points of intersection of the two graphs are: $(-8, 33)$ and $(1, 6)$. Therefore $a = -8; b = 33; c = 1; d = 6$ and $a + b + c + d = \boxed{32}$

8. The Distance travelled by Ron the Mouse is $80 + 150 = 230$ Centimeters. By the Pythagorean Theorem we get that the magnitude of Displacement of Ron the Mouse is 170 Centimeters. therefore the difference of the 2 is $230 - 170 = \boxed{60}$

9. Since all 3 circles are tangent to each other, $\Gamma_1\Gamma_2 = 22$, $\Gamma_1\Gamma_3 = 34$, and $\Gamma_2\Gamma_3 = 36$, Therefore the perimeter of $\Delta\Gamma_1\Gamma_1\Gamma_3$ is $22 + 34 + 36 = \boxed{92}$

10. Let the 6 monuments be A,B,C,D,E, and F. Since each monument has exactly 1 road going in and 1 road going out. A number of interconnected monuments can be regarded as a circular string. For example, for 6 Interconnected monuments, $ABDCFE$ and $BDCFEA$ are the same strings, and we have to count the amount of strings there are. To do this, we find the number of permutations there are and then divide by the number of monuments which is also the number of possible starting points of the string. This is $\frac{n!}{n} = (n-1)!$

    There are a number of ways the monuments can be connected:

    Case 1: 1 String only -

    6 monuments all interconnected: $5! = 120$ combinations

    Case 2: 2 strings -

    2.1: 1 monument connected to itself, 5 interconnected: $C_1^6 0!4! = 144$ Combinations

    2.2: 2 monuments interconnected, 4 monuments interconnected: $C_2^6 1!3! = 90$ Combinations

    2.3 3 monuments interconnected, 3 monuments interconected: $\frac{C_3^6}{2!} 2!2! = 40$ Combinations

    (The Expression above had to be divided by 2! because the two groups of 3 monuments can be reordered)

    Total combinations for Case 2: $144 + 90 + 40 = 274$ Combinations

    Case 3: 3 Strings -

    3.1: 2 monuments interconnected, 2 monuments interconected, 2 monuments interconnected: $\frac{C_2^6 C_2^4}{3!} 1!1!1! = 15$ Combinations

3.2: 1 monument connected to itself, 2 interconnected, 3 interconnected: $C_1^6 C_2^5 0!1!2! = 120$ Combinations

3.3: 1 monument connected to itself, 1 monument connected to itself, 4 monuments interconnected: $\frac{C_1^6 C_1^5}{2!} 0!0!3! = 90$ Combinations

Total combinations for Case 3: $15 + 120 + 90 = 225$ Combinations

Case 4: 4 Strings

4.1: 1 monument connected to itself, 1 monument connected to itself, 1 monument connected to itself, 3 interconnected: $\frac{C_1^6 C_1^5 C_1^4}{3!} 0!0!0!2! = 40$ Combinations

4.2: 1 monument connected to itself, 1 monument connected to itself, 2 interconnected, 2 interconnected: $\frac{C_2^6 C_2^4 C_1^2}{2!2!} 0!0!1!1! = 45$ Combinations

Total combinations for Case 4: $40 + 45 = 85$ Combinations

Case 5: 5 Strings

The Only possible combination of 5 strings to to have 2 monuments interconnected, and the rest are connected to itself. The Number of Combinations for Case 5 would be: $\frac{C_1^6 C_1^5 C_1^4 C_1^3}{4!} 0!0!0!0!1! = 15$ Combinations

Case 6: 6 strings

There is only 1 possible combination for case 6 and that is that each monument is connected to itself

In total there are: $120 + 274 + 225 + 85 + 15 + 1 = \boxed{720}$ Combinations

Long Questions

Explain your answer for each of the questions

1. By the double angle formula for sin we know that: $\sin 2x = 2 \sin x \cos x$, substitute that onto the left hand side, getting:

$$\begin{aligned}\frac{\sin 2x}{\cos^2 x} &= \frac{2 \sin x \cos x}{\cos^2 x} \\ &= \frac{2 \sin x}{\cos x} \\ &= 2 \tan x \\ &= RHS\end{aligned}$$

   Q.E.D

2. By Expanding the brackets:

$$\sqrt{\frac{(a+b)^2 + (a-b)^2}{2}} = c$$

$$\sqrt{\frac{(a^2+2ab+b^2)+(a^2-2ab+b^2)}{2}}=c$$
$$\sqrt{\frac{2a^2+2b^2}{2}}=c$$
$$\sqrt{a^2+b^2}=c$$

Squaring both sides:

$$a^2+b^2=c^2$$

a, b, and c are pythagorean triplets, so therefore a, b, and c are side lengths of a right angled triangle

Q.E.D

3. By expanding $(1+x)(1+y)(1+z)$, we obtain

$$(1+y+x+xy)(1+z)$$

$$=1+y+x+xy+z+yz+xz+xyz$$
$$=1+x+y+z+xy+yz+zx+xyz$$

(Substituting $xy+yz+zx+xyz=2$)

$$=1+x+y+z+2$$

$$\therefore (1+x)(1+y)(1+z)=x+y+z+3$$

The minimum value of $(1+x)(1+y)(1+z)$ is equal to the minimum value of $x+y+z+3$

Because $x,y,z \geq -1$, The minimum of $x+y+z+3$ is:

$$x+y+z+3$$

$$=-1-1-1+3$$
$$=\boxed{0}$$

Q.E.D