# Competition #10 Solutions

The Junior Online Math Olympiad

20th October 2014 - 27th October 2014

Answer Key:

1. 4
2. 3
3. 65
4. 3
5. 24
6. 1
7. 1
8. 63
9. 2
10. 4

## Short Questions

1. By Cauchy Schwarz inequality,
   $4(a^2+2) = (a^2+b^2+c^2)(a^2+1+1) \geq (a^2+b+c)^2$. Since equality holds, we have $b = c = 1$. So, $a = \sqrt{2}$. It is easy to see that this is a solution. Thus, $a+b+c = \sqrt{2}+2$, giving $x+y = \boxed{4}$.

2. The remainder of a number when it is divided by 7 is basically taking (mod 7) of the number, so we have:

$$\begin{aligned}\text{final number} &= 2014^{11}\\ &= 5^{11}\\ &= 5^{6+5}\\ &= 5^{\varphi(7)+5}\\ &= 5^{5}\\ &= 3125\\ &= \boxed{3} \pmod 7\end{aligned}$$

3. We solve this using generating functions. We have 30 marks assured, and so see that we just want the total of 17 numbers to be 113, the numbers which are in the range $0 \leq number \leq 9$ (Because only Adi, Cody and Yan Yau topped, nobody else got a score of 10)

   The answer is co-efficient of $x^{113}$ in the expansion of $\left(\sum_{k=0}^{9} x^k\right)^{17}$, and it is 25783815044387. This has digit sum $2+5+7+8+3+8+1+5+0+4+4+3+8+7 = \boxed{65}$

4. Note that $p^2+2 \equiv 2$ or $0 \pmod 3$, since a square can be only congruent to 0 or 1 mod 3. However, if $p \neq 3$, then $p^2 \equiv 1 \pmod 3$, so $3 \mid p^2+2$, which gives $p = 1$, a contradiction. It is easy to see that $p = 3$ is a solution. Thus, the answer is $\boxed{3}$.

5. Notice $\dfrac{2+3y^2+3x^2(1+y^2)^2+x(6y+6y^3)}{y+x(1+y^2)} = \frac{2}{x+xy^2+y}+3x+3xy^2+3y = \frac{2}{z}+3z$. By AM-GM, $\frac{\frac{2}{z}+3z}{2} \geq \sqrt{6} \Leftrightarrow \left(\frac{2}{z}+3z\right)^2 \geq \boxed{24}$

6. Note that $(n-2)(n^2+n+3) = n(n^2-n+1)-6$, so since $n^2+n+3 = (n+\frac{1}{2})^2+\frac{11}{4} \geq \frac{11}{4} > 2$, if $n > 3$ or $n < 1$, then the number is composite. So, we just need to examine $n = 1, 2, 3$. It is easy to check that only $n = 1$ works. Thus, the answer is $\boxed{1}$.

7. $(a_n+1)(a_{n-1}-1) = -1$ is equivalent to $a_na_{n-1} = a_n - a_{n-1}$, or $\frac{1}{a_{n-1}} - \frac{1}{a_n} = 1$. So, $\frac{1}{a_1} - \frac{1}{a_{2014}} = 2013$, which gives $a_{2014} = \boxed{1}$.

8. $4 = \text{Area of } \triangle ABC = \frac{1}{2}AB \cdot BC \sin \angle ABC = \frac{1}{4}AB \cdot BC$. So, $AB \cdot BC = 16$. Thus, $AC = 3$. Since $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos 30^\circ \Rightarrow AB^2+BC^2 = 16\sqrt{3}+9$, we have $(AB+BC)^2 = 16\sqrt{3}+41$ or $AB+BC = \sqrt{16\sqrt{3}+41}$. So, the perimeter of the triangle is $\sqrt{16\sqrt{3}+41}+3$, and the answer is $16+3+41+3 = \boxed{63}$.

9. The answer is $\boxed{2}$. Aditya ALWAYS wins! He can just keep incrementing until he reaches $[n(n+1)(n+2)...(n+1000)]^3$ when he's at level $n$. It is easy to check that this will take Aditya to whatever level he likes to play until he's frustrated and quits playing.

10. $AD = \frac{AB+AC-BC}{2} = \frac{AC-\frac{AC}{2}}{2} = \frac{AC}{4}$. So, $\frac{AC}{AD} = \boxed{4}$.

## Long Questions

1. By the AM-HM inequality on the 3 numbers (a,b,c) , we have

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Simply multiplying by $\dfrac{3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{a+b+c}$ on both sides, (this is a positive number so inequality won't change), the result is obtained.

2. Notice that $a+b+c=1$. This makes the new roots become $a-\frac{1}{b+c-a-b-c} \leftrightarrow a+\frac{1}{a}$. Substituting $ab+ac+bc=2$ and $abc=3$ by Vieta's Formulae, we have that $a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}=\frac{5}{3}$, $(a+\frac{1}{a})(b+\frac{1}{b})+(a+\frac{1}{a})(c+\frac{1}{c})+(b+\frac{1}{b})(c+\frac{1}{c})=0$ and $(a+\frac{1}{a})(b+\frac{1}{b})(c+\frac{1}{c})=\frac{5}{3}$. Adapting the cubic equation to integer coefficients, we find that it is of the form $k(3x^3-5x^2-5)$, where $k$ is an integer.

3. Just take $a=n+1, b=2n, c=n$ for all positive integers $n$.