Competition #10 Solutions

The Junior Online Math Olympiad

20th October 2014 - 27th October 2014

Answer Key:

1. 1
2. 2035
3. $\frac{167}{1323}$
4. 24
5. 25
6. 11
7. 725
8. 5
9. 1008
10. 90

Short Questions

1. (Melodies) What is the number of real solutions for the equation

$$(x^{10660}+1)(1+x^2+x^4+\ldots+x^{10658}) = 10660x^{10659}?$$

Solution Divide both sides of equation by x^{10659} to yield:

$$(x+\frac{1}{x^{10659}})(1+x^2+x^4+\ldots+x^{10658}) = 10660.$$

Thus, $10660 = x+\frac{1}{x}+x^3+\frac{1}{x^3}+\ldots+x^{10659}+\frac{1}{x^{10659}} \geq 2\cdot 5330$. Equality holds only when $x=\frac{1}{x}=1$ or -1. However, it is clear that $x=-1$ does not satisfy the equation above, thus there is only $\boxed{1}$ real solution, which is $x=1$

2. (Adi) Given that $f(x)$ is a cubic polynomial with real coefficients and $f(1) = 3, f(3) = 8, f(8) = 13, f(13) = 19$. Then $f(26) = \frac{a}{b}$ such that $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Find $a + b$.

   Solution By Lagrange interpolation, the polynomial will be $f(x) = \frac{-3}{168}(x-3)(x-8)(x-13) + \frac{8}{100}(x-1)(x-8)(x-13) - \frac{13}{175}(x-1)(x-3)(x-13) + \frac{19}{600}(x-1)(x-3)(x-8)$

   At $x = 26$, value of this polynomial is $\frac{2021}{14}$. Thus the answer is $\boxed{2035}$.

3. (Adi) There is a cubic die with 6 faces numbered from 1 to 6. Due to the composition of the die, it's not fair one. Probabilitylaugbability of getting 1 be x. We know that the sum of all the probabilities is always 1 , because at least one of them will appear on top. Hence the sum of all probabilities, which are $x, 2x, 3x, 4x, 5x, 6x$ for $1, 2, 3, 4, 5, 6$ respectively, is 1.

$$\implies x + 2x + 3x + 4x + 5x + 6x = 1 \implies x = \frac{1}{21}$$

   Now, the sum 15 can be obtained only by getting the numbers $(4, 5, 6)$ and its 6 permutations or $(6, 6, 3)$ and its 3 permutations or $(5, 5, 5)$.

   According to the rule of sum and rule of product, the probability of getting the sum 15 will be

$$6 \times \frac{4}{21} \times \frac{5}{21} \times \frac{6}{21} + 3 \times \frac{6}{21} \times \frac{6}{21} \times \frac{3}{21} + \frac{5}{21} \times \frac{5}{21} \times \frac{5}{21}$$

$$= \frac{6 \times 4 \times 5 \times 6 + 3 \times 6 \times 6 \times 3 + 5 \times 5 \times 5}{21^3} = \frac{1169}{9261} = \frac{167}{1323}$$

4. (Guilherme) Evaluate the smallest value that the expression $\left(\frac{2 + 3y^2 + 3x^2(1 + y^2)^2 + x(6y + 6y^3)}{y + x(1 + y^2)}\right)^2$ can attain, if x, y are positive numbers.

   Solution

   Notice $\frac{2 + 3y^2 + 3x^2(1 + y^2)^2 + x(6y + 6y^3)}{y + x(1 + y^2)} = \frac{2}{x+xy^2+y} + 3x + 3xy^2 + 3y = \frac{2}{z} + 3z$. By AM-GM, $\frac{\frac{2}{z}+3z}{2} \geq \sqrt{6} \Leftrightarrow \left(\frac{2}{z} + 3z\right)^2 \geq \boxed{24.}$

5. (Guilherme) The number 1 is a double root in the equation $x^4 - 2x^3 - 3x^2 + ax + b = 0$. Evaluate the digit sum of b^a.

   Solution

   By Briot-Ruffini's algorithm, we ahve $b + a - 4 = 0$ and $a - 8 = 0$, which leads us to $a = 8,\ b = -4$. Thus, $b^a = (-4)^8 = 65536$, and its digit sum is $\boxed{25}$

6. (Guilherme) Consider the triangle ABC on the cartesian plane, with vertices $A = (0; 0)$, $B = (3; 4)$, $C = (8; 0)$. The rectangle $MNPQ$ has its vertices M, N on the x-axis, and the vertices Q and P are on the sides $\overline{AB}$ and $\overline{BC}$, respectively. Among all rectangles built this way, the one that has maximum area is the one in which the point P is located at $(p_x; p_y)$. Evaluate $p_x \cdot p_y$.

   Solution

   Let $\overline{QP} = b$ and the distance from B to $\overline{QP} = h$. By triangle similarity ($\triangle BPQ \equiv \triangle BCA$), we have $\frac{b}{h} = \frac{8}{4} \Leftrightarrow b = 2h$. To maximize the area $b(4-h) = 2h(4-h)$, we found its maximum is achieved at $h = 2$. The point of height $p_y = 2$ in the line $\overline{BC}$ happens at $p_x = \frac{3+8}{2} = \frac{11}{2}$, and thus $p_x \cdot p_y = \boxed{11}$

7. (ZS) Let $ABCD$ be a quadrilateral such that it's diagonals intersect at E. $AE = 1, BE = 2, AD = 4, CE = 8$. DE has integer length. $AB \cdot CD = 25$. Find the value of $100BC$.

   Solution

   Note that by Triangle inequality and the hypothesis, we have $DE = 4$, since $3 < DE < 5$ and DE is a positive integer. This gives $AE \cdot CE = BE \cdot DE = 8$, so by the converse of Power of Point, $ABCD$ is cyclic. By Ptolemy's Theorem, $AB \cdot CD + BC \cdot AD = AC \cdot BD = 9 \cdot 6 = 54$. So, $BC \cdot AD = 54 - AB \cdot CD = 54 - 25 = 29$. Since $AD = 4$, $BC = \frac{29}{4}$. So, $100BC = 29 \cdot 25 = \boxed{725}$.

8. (ZS) Let ABC be a triangle such that $AC = 2BC$. The angle bisector of $\angle ACB$ meets AB at F. E is a point on AC such that $AE = 3EC$. Let BE intersect CF at G. AG intersects BC at D. Find $\frac{AG}{GD}$.

   Solution

   By angle bisector theorem, $\frac{AF}{FB} = \frac{AC}{BC} = 2$. So, by van Aubel's theorem, $\frac{AG}{GD} = \frac{AF}{FB} + \frac{AE}{EC} = 2 + 3 = 5$.

   Sidenote: Proof of van Aubel's theorem

   Applying Menalaus on $\triangle ABD$ and $\triangle ACD$ gives

   $\frac{AF}{FB} \cdot \frac{BC}{CD} \cdot \frac{DG}{GA} = 1$ and $\frac{AE}{EC} \cdot \frac{CB}{BD} \cdot \frac{DG}{GA} = 1$.

   So, $\frac{AF}{FB} = \frac{CD}{BC} \cdot \frac{AG}{DG}$ and $\frac{AE}{EC} = \frac{BD}{BC} \cdot \frac{AG}{DG}$. Adding these two equations give $\frac{AF}{FB} + \frac{AE}{EC} = \frac{AG}{DG}(\frac{BD+DC}{BC}) = \frac{AG}{DG}$.

9. Let $f(x) = \frac{x^2}{2x^2-2x+1}$. Determine the value of $f(\frac{1}{2015}) + f(\frac{2}{2015}) + ... + f(\frac{2015}{2015})$.

   Solution

   Note that $f(x) = f(1-x)$, so the required sum is just $1007 + f(1) = 1007 + 1 = 1008$.

10. (ZS) Let ABC be a triangle and let Y be on ray CB such that $CB = 2YB$. Let X be on ray AY such that $XY = AY$. Suppose $AC = 2BX$. Find, in degrees, $\angle CTA$, where T is the midpoint of BX.

Solution

Y is the midpoint of AX and since $CB = 2YB$, B is the centroid of $\triangle CAX$. Thus, XB intersects AC at M, the midpoint of AC. Thus, $TM = BX = \frac{AC}{2} = AM = CM$. Thus, T lies on the circle with diameter AC. This implies $\angle CTA = 90°$.

Long Questions

1. (ZS) Prove that for all positive real numbers a, b, c, such that $a^2+b^2+c^2 = 2(ab+bc+ca)$, $\displaystyle\sum_{cyc} \frac{(a+b)^2}{b^2+c^2} \geq 4$.

Solution

By Titu's lemma,

$$\sum_{cyc} \frac{(a+b)^2}{b^2+c^2} \geq \frac{4(a+b+c)^2}{2(a^2+b^2+c^2)} = 2 + \frac{4(ab+bc+ca)}{a^2+b^2+c^2} = 4.$$

2. Define $M(x)$ as the monic polynomial with integer coefficients of least degree such that $M(x) = 0$. For instance, $M(3) = x - 3$, $M(\sqrt{2}) = x^2 - 2$ and $M(2 - \sqrt{2}) = x^2 - 4x + 2$. Find, with proof, $M(\sqrt[3]{3} - \sqrt[3]{2})$.

Solution

Setting $x + \sqrt[3]{2} = \sqrt[3]{3}$ and cubing both sides, we have $3\sqrt[3]{2}x(x + \sqrt[3]{2}) = 1 - x^3$. Substituting and cubing both sides, we find $x^9 - 3x^6 + 165x^3 - 1 = 0 = M(\sqrt[3]{3} - \sqrt[3]{2})$.

3. Show that, for some positive integer values of $(a_1, a_2, \cdots, a_n, b_n)$, the condition $\sum_{i=1}^{n} a_i = {b_n}^2$ can always be attained for integer values of $n \geq 2$.

Solution

Consider $n = 2$ and $a_1 = 3, a_2 = 4$. We clearly have that ${b_2}^2 = 25$, and thus (a_1, a_2, b_2) is a Pythagorean triple. Letting $a_3 = 12$, we have $that(b_2, a_3, b_3)$ is also a Pythagorean triple, and this goes on for (b_3, a_4, b_4) when $a_4 = 84$, and this can continue endlessly. Thus, for all integer values of $n > 1$, the sum of some squares can be another square.