Competition #12 Solutions

The Junior Online Math Olympiad

22nd December 2014 - 29th December 2014

Answer Key:

1. 51
2. 96
3. 2014
4. 1
5. 0
6. 24
7. 8
8. 25
9. 1
10. 84

Short Questions

1. (Aditya) Given that

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}}=\frac{101061}{14254}$$

and all of a, b, c, d, e are **positive integers**, then find $a+b+c+d+e$.

Solution

See that $\dfrac{101061}{14254}=7+\frac{1283}{14254}$. Thus $a=7$.

The fractional part is actually the reciprocal of $b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}$

$\frac{14254}{1283} = 11 + \frac{141}{1283}$. Thus $b = 11$.

Similarly, $\dfrac{1283}{141} = 9 + \frac{14}{141}$. Thus $c = 9$.

Similarly, $\dfrac{141}{14} = 10 + \frac{1}{14}$. Thus $d = 10, e = 14$.

Hence $a + b + c + d + e = 7 + 11 + 9 + 10 + 14 = \boxed{51}$

2. (Navi) Let positive reals a, b, c satisfy $abc = 1$. Find the last 3 digits of the minimum value of

$$(a + 2015b)(b + 2015c)(c + 2015a)$$

Solution Note that

$$(a + 2015b) \geq 2016 \sqrt[2016]{ab^{2015}}$$

and similarly

$$(b + 2015c) \geq 2016 \sqrt[2016]{bc^{2015}}$$

$$(c + 2015a) \geq 2016 \sqrt[2016]{ca^{2015}}$$

Multiplying the 3 inequalities we get

$$(a+2015b)(b+2015c)(c+2015a) \geq 2016 \sqrt[2016]{ab^{2015}} \cdot 2016 \sqrt[2016]{bc^{2015}} \cdot 2016 \sqrt[2016]{ca^{2015}} = 2016^3 abc = 2016^3$$

So the minimum is 2016^3, which is obtainable by setting $a = b = c = 1$. $2016^3 = 8193540096$, hence its last three digits are $\boxed{096}$

3. (Navi) Let N be a square-free integer, with divisors $1 < d_2 < \cdots < d_8$. Given that $d_4 = 2d_3$ and $d_7 - d_6 = 901$, find N.

Solution: First, note that d_2 must be a prime. As N is square-free, then d_3 is another prime (as it cannot be any power of d_2). From the first condition, $d_4 = 2d_3$, and therefore d_4 is composite. Hence $d_4 = d_2 d_3 \Rightarrow d_2 = 2$. Consequently, d_5 must be largest prime divisor (otherwise d_5 is not square-free, imply N is not square-free)

From the second condition, we have $d_7 - d_6 = d_5(d_3 - d_2) = d_5(d_3 - 2) = 901 = 17 \cdot 53$. Since $d_5 > d_3$, we must have $d_5 = 53$, and $d_3 - 2 = 17 \Rightarrow d_3 = 19$. This gives $N = d_2 d_3 d_5 = 2 \cdot 19 \cdot 53 = \boxed{2014}$.

4. (Navi) Let ABC be a triangle and let two points P, Q lie on segment BC so that Q is closer to B than P. Let circumcircle of APC intersect line AB at D and intersect AQ at E, and BE intersect circumcircle of DPB and APC again at K and L respectively. Let $M = DK \cap AQ$. Given that $CL \parallel AB$, find $\frac{ME}{MK}$.

Solution

Note that

$$\angle MEK = \angle DEB = \angle DPB = \angle A = \angle ACL = \angle AKL = \angle MKE$$

so $\triangle MKE$ is isoceles, so $\frac{ME}{MK} = \boxed{1}$.

5. (Navi) Find the number of ordered quadruples (w, x, y, z), where w, x, y, z are non-negative reals, that satisfy the equation

$$(\frac{1}{w}+\frac{1}{x}+\frac{1}{wx})(\frac{1}{y}+\frac{1}{z}+\frac{1}{yz})+4=(2+\frac{1}{wxyz})$$

.

Solution

Make the substitution, $w = a^2, x = b^2, y = c^2, z = d^2$, we get

$$(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{a^2b^2})(\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{c^2d^2})+4=(2+\frac{1}{(abcd)^2})$$

where a, b, c, d are nonzero reals. Multiplying $a^2b^2c^2d^2$, We get

$$(a^2+b^2+1)(c^2+d^2+1)=(2abcd+1)^2-(2abcd)^2=4abcd+1$$

By completing the squares,

$$(ac-bd)^2+(ad-bc)^2+a^2+b^2+c^2+d^2=0$$

Since squares are non-negative, $a = b = c = d = 0 \Rightarrow (w, x, y, z) = (0, 0, 0, 0)$, However, this solution involves the division of 0, hence there are actually no solutions, and the answer of $\boxed{0}$

6. (Melodies) Find the remainder when $(5 \cdot 15122014^{2014} + 1)(5 \cdot 15122014^{2014} + 2)(5 \cdot 15122014^{2014} + +3)(5 \cdot 15122014^{2014} + 4)$ is divided by 25.

   Solution

   For all $n \in \mathbb{Z}$, the following holds: $(5n + 1)(5n + 2)(5n + 3)(5n + 4) \equiv \boxed{24} \pmod{25}$. Simply expand to prove.

7. (Guilherme) Let ψ be a complex number such that $(1 + 2i)^3 + (3 + 2i)^1 + \psi = 0$. Evaluate ψ 's real part.

   Solution

   Expanding $(1 + 2i)^3$, we get $-11 + 2i$. Hence the expression becomes $-11 - 2i + 3 + 2i + \psi = 0$. The reals add up to -8, since the total sum adds up to 0, the real part of ψ is $\boxed{8}$

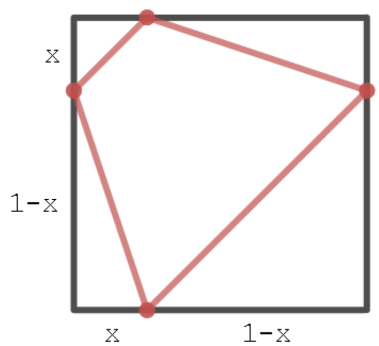
8. (Guilherme) Consider the $\mathbb{N} \to \mathbb{R}_+$ function $f(x)$ such that the sum of the first n values of $f(x)$, starting at 1, is $\sqrt{n}$. Find the integer part I_p of the sum of the first 25 values of $\frac{1}{f(x)}$, starting at 1. As an example of nomenclature $I_p(2 + \sqrt{5}) = 2,\ I_p(7) = 7,\ I_p(\pi) = 0$.

   Solution

   If the sum of the first n values of $f(x)$, starting at 1, is $\sqrt{n}$, we can deduce that $f(x) = \sqrt{x} - \sqrt{x-1}$ is a satisfying function, since its consecutive sum is telescoping.

Furthermore, we have $\dfrac{1}{f(x)} = \sqrt{x} + \sqrt{x-1}$, and the sum of the first 25 values of $\dfrac{1}{f(x)}$, starting at 1, can be written as $2(1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{24}) + \sqrt{25}$. The integers in this expression are the square roots of perfect squares, adding up to $2(1+2+3+4)+5$, so the integer part of the sum is $I_p = \boxed{25}$.

9. (Guilherme)For $0 < x < 1$, let S_{red} be the area inside the red trapezoid, and S_{black} be the area between the black square and the red trapezoid. For $x = \dfrac{1}{\pi}$, find the sum of the first ten decimal places of the ratio between S_{red} and S_{black}, plus the integer part.



Solution

The value of S_{black} can be written, in function of x, as the sum of the areas of the four triangles that have black and red sides.

Algebraically, $\frac{x^2}{2} + 2 \cdot \frac{x(1-x)}{2} + \frac{(1-x)^2}{2} = \dfrac{1}{2}$. This leads to the result that S_{red} is also $\dfrac{1}{2}$, and so the desired area ratio is 1. The sum of the first ten decimal places of the ratio is zero, and adding the integer part leads to the answer $\boxed{1}$.

10. (Guilherme) P and Q be two-digit numbers such that the last three digits of P^2 and Q^2 are respectively 929 and 464. Evaluate the last three digits of PQ.

Solution

Let $P = \overline{ab} = 10a + b$ and $Q = \overline{cd} = 10c + d$.

Upon analysis of the last digit of P, we can conclude that $b = 3$ or 7, since the last digit of both, when squared, yields $\cdots 9$. Letting $b = 3$ and testing $a = 1 \to 9$, we find out that there is no possible value for P on these conditions, so $b = 7$. By testing a in its range again, we find that $a = 7$ yields $P^2 = 5929$, with the three last digits as desired.

Similarly, analysis leads us to $d = 2$ or 8. Upon selecting $d = 2$, $c = 9$ returns $Q^2 = 8464$, with the three last digits as desired. Testing $d = 8$, however, does not return a possible value for Q under the conditions.

Multiplying 77 by 92, we find $PQ = 7084$, so the final answer is 084 or $\boxed{84}$

Long Questions

1. A quadrilateral $ABCD$ has side lengths $AB = 5, BC = 6, CD = 8, DA = 7$. Prove that there exists a point P in $ABCD$ such that the perpendiculars from P to the sides of $ABCD$ are equal.

   (2 Points)

   Solution

   Since $13 = AB + CD = BC + DA$, so $ABCD$ has an incircle. We take P to be the center of the incircle which satisfies the properties.

2. (Navi) Do there exist positive integers x, y such that $\sqrt{5^x + 7^y}$ is an integer?

   (2 points)

   Solution

   Suppose $\sqrt{5^x + 7^y}$ is an integer for some positive integers x and y, let $A = 5^x + 7^y$, then A is a perfect square.
   By looking at the last 2 digits of 5^x+7^y, the possible remainders modulo 100, are $6, 12, 26, 32, 48, 54, 68, 74$.
   $(5^x \equiv 5, 25 \pmod{100}, 7^y \equiv 7, 49, 43, 1 \pmod{100})$

   - If the remainders are $6, 26, 54, 74$, 2 divides A but 4 do not divides A, so A is not a perfect square.

   - If the remainders are $12, 32, 48, 68$, the perfect square A ends with 2 or 8, a contradiction.

   So A is never a perfect square, thus $\sqrt{5^x + 7^y}$ is never an integer for all positive integers x and y.

3. (ZS) Prove that $3^{2^n} - 1$ can be written as the sum of two squares for all positive integers n.

   (3 points)

   Solution

   $3^{2^n} - 1 = (2^2 + 2^2)(3^2 + 1^2)(3^4 + 1^2)(3^8 + 1^2)...(3^{2^{n-1}} + 1^2)$, which is the product of sum of two squares.
   Thus, 3^{2^n} is also the sum of two squares.

   Sidenote: The product of two sum of 2 squares can be written as the sum of 2 squares.

   $(a^2 + b^2)(c^2 + d^2) = (ad - bc)^2 + (ac + bd)^2$.