Competition #2 Solutions

The Junior Online Math Olympiad

20th January 2014 - 27th January 2014

Answer Key:

1. 1
2. 9
3. 11
4. 8
5. 4
6. 5
7. 30
8. 11
9. 7
10. 1

Short Questions

1. 6^n ends in 6, so $6^n - 1$ will end in a 5, All number that end in 5 except 5 are composite. Therefore there is only $\boxed{1}$ interger n such that $6^n - 1$ is prime and it is $6^1 - 1 = 5$

2. For all integers k:
$$16 + 13k \equiv 3 \pmod{13}$$

$$\therefore \left(\prod_{k=0}^{30}(16+13k)\right) \equiv 3^{30}$$
$$\equiv \left(3^3\right)^{11}$$
$$\equiv 27^{10} \times 3$$
$$\equiv 1^{10} \times 3$$
$$\equiv 3 \pmod{13}$$

For all integers i:

$$140 - 13i \equiv 10 \pmod{13}$$

$$\therefore \left(\sum_{i=0}^{10}(140-13i)\right) \equiv 10(11)$$
$$\equiv 110$$
$$\equiv 6 \pmod{13}$$

Combining these two results:

$$\left(\prod_{k=0}^{30}(16+13k)\right) + \left(\sum_{i=0}^{10}(140-13i)\right) \equiv 3+6$$
$$\equiv \boxed{9} \pmod{13}$$

3. Alfred goes at a constant velocity of 7 km h^{-1}. Since his velocity does not change in the course of the race he will finish the race in 10 hours. Boris goes at 10 km h^{-1} at the beginning and then goes at 5 km h^{-1} after he trips. Let d be the distance he runs before he trips. His total time will be:

$$\text{Time of Boris} = \frac{d}{10\text{kmh}^{-1}} + \frac{70\text{km} - d}{5\text{kmh}^{-1}}$$
$$= \frac{d + 2(70\text{km} - d)}{10\text{kmh}^{-1}}$$
$$= \frac{140\text{km} - d}{10\text{kmh}^{-1}}$$

If Boris' time is more than 10 hours then this will mean that Alfred won the race.

First we should calculate the distance from the start where Boris' time will be exactly 10 hours, then if d is less than that distance, Alfred will win.

Substitute 10 hours as the time of Boris:

$$\begin{aligned} 10\text{h} &= \frac{140\text{km} - d}{10\text{kmh}^{-1}} \\ 100\text{km} &= 140\text{km} - d \\ d &= 40\text{km} \end{aligned}$$

If $d < 40$km, then Alfred will win. d has a range of $0\text{km} \leq d \leq 70\text{km}$, so therefore the chance that Alfred will win the race is $\frac{40\text{km}}{70\text{km}} = \frac{4}{7}$. Therefore $a = 4$ and $b = 7$, so $a + b = \boxed{11}$

4. Add the second and third equations getting:

$$\begin{aligned} 4x^2 + 4xy + y^2 &= z^2 \\ (2x + y)^2 &= z^2 \\ 2x + y &= z \end{aligned}$$

Substituting $2x + y = z$ into the first equation and solve for y:

$$\begin{aligned} 2x + y &= 2x + 4y + 5 \\ 3y &= -5 \\ y &= -\frac{5}{3} \end{aligned}$$

So $a = 5$; $b = 3$. Therefore $a + b = \boxed{8}$

5. Substitute $y = \sqrt[4]{-6x^2 + 16}$ into $x^4 + y^4 = 8$ and solve for x:

$$x^4 + \left(\sqrt[4]{-6x^2 + 16}\right)^4 = 8$$

$$x^4 + -6x^2 + 16 = 8$$

$$x^4 + -6x^2 + 8 = 0$$

$$(x^2 - 2)(x^2 - 4) = 0$$

$$x^2 = 2 \text{ or } x^2 = 4$$

$$x = \sqrt{2} \text{ or } -\sqrt{2} \text{ or } 2 \text{ or } -2$$

However, in the equation $y = \sqrt[4]{-6x^2+16}$, $\sqrt[4]{-6x^2+16}$ cannot be negative. So x cannot be 2 or -2.

The sum of the squares of the x co-ordinates is:

$$(\sqrt{2})^2 + (-\sqrt{2})^2 = \boxed{4}$$

6.

$$5^{2x} + 5^5 = 5^{x+3} + 5^{x+2}$$
$$5^{2x} + 5^5 = 5\left(5^{x+2}\right) + 5^{x+2}$$
$$5^{2x} + 5^5 = 6\left(5^{x+2}\right)$$
$$5^{2x} - 6\left(5^{x+2}\right) + 5^5 = 0$$
$$5^{2x} - 6\left(5^2 5^x\right) + 5^5 = 0$$
$$5^{2x} - 150\left(5^x\right) + 5^5 = 0$$
$$5^{2x} - (125+25)\left(5^x\right) + 5^5 = 0$$
$$\left(5^x\right)^2 - \left(5^3+5^2\right)\left(5^x\right) + 5^5 = 0$$
$$\left(5^x - 5^3\right)\left(5^x - 5^2\right) = 0$$
$$5^x = 5^2 \text{ or } 5^x = 5^3$$
$$x = 2 \text{ or } x = 3$$
$$\therefore \text{ the sum of the solutions of } x \text{ is: } 2+3 = \boxed{5}$$

7. First you choose 3 dalmatians out of the 15 and put them in the first cage, which is $\binom{15}{3}$, then you choose 3 more dalmatians out of the 12 left and put them in the second cage, which is $\binom{12}{3}$, and so on, then you multiply these combinations together, obtaining:

$$\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 168168000$$

The sum of the digits of 168168000 is equal to $\boxed{30}$

8. By the Angle Bisector theorem, we get that:

$$\frac{AB}{BD} = \frac{AC}{CD}$$

Manipulating the equation we get:

$$\frac{AB}{AC} = \frac{BD}{CD} = \frac{10}{12} = \frac{5}{6}$$

Hence $p = 5$, $q = 6$, and $p + q = 5 + 6 = \boxed{11}$

9. The Probability that the dart hits the Second Region is the area of the Second region over the area of the whole dart board. The Area of the second region is $x^2\text{cm}^2 - 2^2\text{cm}^2$ and the area of the whole dart board is 81cm^2. We know that the probability is $\frac{5}{9}$, so:

$$\begin{aligned} \frac{\left(x^2-4\right)\text{cm}^2}{81\text{cm}^2} &= \frac{5}{9} \\ \frac{x^2-4}{81} &= \frac{5}{9} \\ x^2-4 &= 45 \\ x^2 &= 49 \\ x &= \boxed{7} \end{aligned}$$

10. Since $|x|$ is the answer to Question 10 and we are asked to find the value of $\frac{1}{x}$, we have the equation $|x| = \frac{1}{x}$ and we ahve to solve for x

Multiplying both sides by x we get:

$$x|x| = 1$$

If x is negative, $|x|$ is positive so $x|x|$ is negative, since $x|x| = 1$ we know that x cannot be negative, therefore x is positive.

If x is positive then $|x| = x$, so we get:

$$x \times x = 1$$

$$x^2 = 1$$

$$x = 1$$

Therefore $|x| = \frac{1}{x} = \boxed{1}$

Long Questions

1. All numbers whose digits are a permutation of 1234567890 have a digit sum of 45. This means all numbers whose digits are a permutation of 1234567890 are divisible by 9. Therefore all numbers whose digits are a permutation of 1234567890 are not prime as one of their factors is 9.

2. Let Ronald the Mouse moving upwards be denoted by u and Romeo the Mouse moving rightwards be denoted by r. A Path for Ronald the Mouse to get to the rightmost corner of the $n \times n$ grid will be a string that includes n u's and n r's. The number of possible combinations of strings will be:

$$\frac{(2n)!}{n!n!}$$

This is because there are $2n$ letters in the string (n u's and n r's) so there are 2n! combinations to organise a 20 letter string but since the u's and the r's repeat n times you have to divide it by $n!$ twice, once for the n u's and once for the n r's.

The above expression is equivalent to $\binom{2n}{n}$, therefore we have proved that the number of paths Ronald the Mouse can take will always be in the form $\binom{2n}{n}$ for any $n \times n$ grid Q.E.D

3. Multiply the expression by 4 and then complete the square, we get:

$$(y+5)^2 + (2x+3)^2 + (4x-y)^2 \geq 0$$

Since the square of a real number is always positive, the above statement is always true for real values of x and y. Hence Proved.

Q.E.D.