Competition #3 Solutions

The Junior Online Math Olympiad

20th January 2014 - 27th January 2014

Answer Key:

1. 9
2. 906
3. 400
4. 19
5. 210
6. 12
7. 625
8. 1
9. 8
10. 330

Short Questions

1. Since $27 = 3^3$ and $8 = 2^3$, we have $\left(2^3\right)^m = 3^3$, so $2^m = 3$. Squaring both sides leads to our answer of $\boxed{9}$.

2. $\frac{20}{14}$ is $\frac{10}{7}$. After the decimal point is a recurring sequence of length 6, namely 428571. $\frac{2014}{6}$ is $335\frac{2}{3}$. So we're looking for 4+2+8+5+7+1, which is 27. 27×335, added to the first 2/3 of the sequence, 9045+19=9064, the first three digits of this number is $\boxed{906}$.

3. There are a number of ways these scenes can be chosen.

   Case 1: All the groups choose distinct scenes

   Since there are 6 groups and 8 movies, You first choose 6 movies out of 8 then you order them as each group is distinct. This is ${}_8P_6 = \frac{8!}{(8-6)!} = 20160$

   Case 2: 2 of the groups choose the same scene and the other groups choose distinct scenes

   First out of the 6 groups, 2 groups are chosen. This is $\binom{6}{2}$. Then you choose 5 scenes out of 8 and then you order them as each group is distinct. This is ${}_8P_5$. The number of possible combinations for Case 2 is the product of the two above numbers which is: $\binom{6}{2}{}_8P_5 = 100800$

   Case 3: 2 of pairs groups choose the same scene and the other groups choose distinct scenes

   First out of the 6 groups 2 groups are chosen to form the first pair, then out of the remaining 4 groups 2 groups are chosen to form the second pair. There will be 4 distict scenes that are chosen, so you choose 4 scenes out of 8 and then you order them as each group is distinct. This is $\binom{6}{2}\binom{4}{2}{}_8P_4 = 151200$

   Case 4: 3 of pairs groups choose the same scene.

   First out of the 6 groups 2 groups are chosen to form the first pair, then out of the remaining 4 groups 2 groups are chosen to form the second pair. The remaining 2 groups will be the 3rd pair so there is no need to choose the 3rd pair. There will be 3 distict scenes that are chosen, so you choose 3 scenes out of 8 and then you order them as each group is distinct. This is $\binom{6}{2}\binom{4}{2}{}_8P_3 = 30240$

   The total number of ways these movies can be chosen will be the sum of the totals of the 4 cases. This is $20160+100800+151200+30240 = 302400$

   The last 3 digits of 302400 is $\boxed{400}$

4.
$$1000N = 101.10101...$$
$$100000N = 10110.10101...$$

   So;

$$\begin{aligned} 100000N - 1000N &= 10110.10101... - 101.10101... \\ 99000N &= 10009 \\ N &= \frac{10009}{99000} \end{aligned}$$

   $\therefore a = 10009; b = 99000$

   $a + b = 109009$

   The digit sum of $a + b$ is $\boxed{19}$

5. The rooks have to be in different rows and different columns. Since there are 6 rooks but 10 rows and columns. We just have to choose 6 rows out of the 10 and 6 columns out of the 10. This is:

$$\binom{10}{6}\binom{10}{6}$$

However, the rooks are also distinct, therefore we have to multiply this number by 6!

Therefore

$$\begin{aligned} n &= \binom{10}{6}\binom{10}{6}6! \\ &= 31752000 \end{aligned}$$

So the product of the non-zero digits of n is: $3 \times 1 \times 7 \times 5 \times 2 = \boxed{210}$

6.

$$100 = 2^2 \times 5^2$$
$$100^n = 2^{2n} \times 5^{2n}$$

Since $5 > 2$, there will be more factors of 2 in 100! than factors of 5. Therefore we should count the number of factors of 5 in 100!

$$\left\lfloor \frac{100}{5} \right\rfloor = 20$$

$$\left\lfloor \frac{100}{25} \right\rfloor = 4$$

There are $20 + 4 = 24$ factors of 5 in 100!, hence the largest n such that $100^n | 100!$ is $\boxed{12}$

7. Let $P(x)$ be the expression. Then, we know $P(x) + P(-x) = 2$ times all the even powers of x. This is because when $-x$ raised to an even power it has the same value as x raised to that power and when $-x$ is raised to an odd power it is equal to -1 times x raised to that power.

By substituting $x = 1$, we get the sum of the coefficients of the function

So we get:

$$\frac{P(1) + P(-1)}{2} = \frac{5^8 + (-5)^8}{2} = 5^8 = 390625$$

The last 3 digits of 390625 is $\boxed{625}$

8. Let O be the center of the larger circle and the centers of the 4 smaller circles be A, B, C, and D respecively.

   Note that $ABCD$ is a square with side length $2r$ and O is the center of $ABCD$.

   The diagonal of the square, AC is $2\sqrt{2}r$. O is the midpoint of AC, so $OA = OC = r\sqrt{2}$

   Extend line segment OC such that it intersects the larger circle, call the point of intersection I.

   Since OI is the radius of the circle, $OI = 1$. Since $OI = OC + CI$, and CI is the radius of the smaller circle (r), so we have:

$$\begin{aligned} 1 &= r\sqrt{2} + r \\ r(\sqrt{2}+1) &= 1 \\ r &= \frac{1}{1+\sqrt{2}} \end{aligned}$$

   Rationalising the denominator:

$$\begin{aligned} r &= \frac{1}{1+\sqrt{2}} \\ &= \frac{1}{1+\sqrt{2}}\frac{\sqrt{2}-1}{\sqrt{2}-1} \\ &= \frac{\sqrt{2}-1}{1} \\ &= \sqrt{2}-1 \end{aligned}$$

   Therefore:

$$\begin{aligned} r(r+2) &= (\sqrt{2}-1)(\sqrt{2}-1+2) \\ &= (\sqrt{2}-1)(\sqrt{2}+1) \\ &= \boxed{1} \end{aligned}$$

9. Since the smaller circle is inscribed in the small square, the center of the circle is the center of the square. Hence the circle is tangent to the midpoints of the sides of the squares. Since the radius of the smaller circle is 1, the side length of the small square is twice the radius of the smaller circle. Hence the side length of the smaller square is 2.

   The radius of the larger circle will be the length from the center of the smaller square to the vertex of the smaller square, which is half the length of the diagonal of the smaller square. So the radius of the larger square is $\frac{2\sqrt{2}}{2} = \sqrt{2}$.

Using the same method as above, the side length of the larger square is twice the length of the radius of the larger circle. Therefore the side length of the larger square is $2\sqrt{2}$, the area of the larger square would be $\left(2\sqrt{2}\right) = \boxed{8}$

10.

$$\begin{aligned}\cos 2x + 2\sin x - \sqrt{3}\sin x &= 1 - \sqrt{3}\\ 1 - 2\sin^2 x + 2\sin x - \sqrt{3}\sin x &= 1 - \sqrt{3}\\ -2\sin^2 x + (2 - \sqrt{3})\sin x + \sqrt{3} &= 0\end{aligned}$$

Let $\sin x = u$

$$\begin{aligned}-2u^2 + (2 - \sqrt{3})u + \sqrt{3} &= 0\\ -(2u + 3)(u - 1) &= 0\\ \therefore u &= -\frac{\sqrt{3}}{2} \text{ or } \qquad u = 1\\ \sin x &= -\frac{\sqrt{3}}{2} \text{ or } \quad \sin x = 1\end{aligned}$$

Because $x \in [0°, 270°]$,

$$x = 240° \text{ or } x = 90°$$

Therefore the sum of the solutions for x is equal to $240 + 90 = \boxed{330}$

Long Questions

1. We will prove this by induction

Note that since ρ is a root of the equation: $x^2 - 2x - 1 = 0$,

$$\rho^2 = 2x + 1$$

First we prove $\rho^n = F(n-1) + F(n)\rho$ is true when $n = 1$

$$\rho^1 = \rho = 0 + 1\rho = F(0) + F(1)\rho$$

Let $\rho^n = F(n-1) + F(n)\rho$ be true when $n = k$ (1), when $n = k + 1$:

$$\begin{aligned}\rho^{k+1} &= \rho^k \times \rho\\ &= \rho(F(k-1) + F(k)\rho)\\ &= F(k-1)\rho + F(k)\rho^2\\ &= F(k-1)\rho + F(k)(2\rho + 1)\\ &= F(k-1)\rho + F(k) + 2F(k)\rho\\ &= F(k) + [F(k-1) + 2F(k)]\rho\\ &= F(k) + F(k+1)\rho\\ &= F((k+1) - 1) + F(k+1)\rho\end{aligned}$$

We have proved $\rho^n = F(n-1) + F(n)\rho$ is true when $n = 1$ and $\rho^n = F(n-1) + F(n)\rho$ is true when $n = k+1$ if it is true for k, therefore we have proven that $\rho^n = F(n-1) + F(n)\rho$ is true for all integers n

2. Since $f(x)$ is a monic polynomial, $f(x)$ can be expressed as:

$$f(x) = (x-\alpha)(x-\beta)(x-\gamma)$$

When $x = -1$:

$$f(-1) = (-1-\alpha)(-1-\beta)(-1-\gamma)$$

$$f(-1) = -(1+\alpha)(1+\beta)(1+\gamma)$$

$$\therefore -f(-1) = (1+\alpha)(1+\beta)(1+\gamma)$$

$$-((-1)^3 - 5(-1)^2 + 6(-1) + 1) = (1+\alpha)(1+\beta)(1+\gamma)$$

$$(1+\alpha)(1+\beta)(1+\gamma) = -(-1-5-6+1)$$

$$(1+\alpha)(1+\beta)(1+\gamma) = \boxed{11}$$

3.

$$\begin{aligned}\sin^4(x) - \cos^4(x) &= \left(\sin^2(x) - \cos^2(x)\right)\left(\sin^2(x) + \cos^2(x)\right)\\ &= \sin^2(x) - \cos^2(x)\end{aligned}$$