Competition #4 Solutions

The Junior Online Math Olympiad

28th April 2014 - 5th May 2014

Answer Key:

1. 5
2. 320
3. 0
4. 121
5. 168
6. 86
7. 77
8. 30
9. 358
10. 17

Short Questions

1. $36 = 2^2 3^2$

   x needs to be divisible by 4 and 9

   The divisibility test for 9 is that the sum of a number's digits needs to be divisible by 9 for the number itself to be divisible by 9.

   1+2+3+4+5+6+7+8=36 and 36 is divisible by 9, hence all elements in S are divisible by 9.

   Because $4 \mid 100$, the last 2 digits of x needs to be divisible by 2.

There are 25 multiples of 4 from 00 to 99. However x has to have no repeated digits and cannot have the digit 9 or 0, so 00, 04, 08, 92, 96, 20, 40, 60, 80, 44, 88 are not possible endings for x. Hence there are $25 - 11 = 14$ possible endings for x.

The remaining 6 digits can be ordered in any way and the conditions will still be satisfied, hence there are $6! \times 14$ different permutations of the digits 12345678 such that the number is divisible by 36.

There are a total of 8! elements in S. Hence the probability that x is divisible by 36 is $\frac{6! \times 14}{8!} = \frac{1}{4}$. Hence $m + n = \boxed{5}$

2. Since there are 3 phrases in a verse and 4 verses. There are 12 phrases in the whole song. There are 10 phrases that Ron wants to put in the song, hence there will be exactly 1 phrase that repeats twice or 2 different phrases will each repeat once.

   Case 1: Exactly 1 phrase repeats twice. It is impossible for the phrases "Ronnam Ronnam Nowy Nowy Ronnam Wisda Taichi!", "Eh Eh", or "Chhhhhikuuuuuuuuuun" to repeat twice, This is because each of these phrases has to appear at least once and these phrases must appear at the end of a verse. However there are only 4 verses so this is impossible.

   Hence the verse that repeats twice must be one of the other verses.

   First we order the number of possible endings. Since there are only 3 endings but 4 verses, 1 of the endings will be any phrase. We will leave this blank. Considering the blank to be an element as well we have $4! = 24$ combinations for endings.

   For the other 9 phrases, there are 7 distinct phrases where 1 of them repeats twice. There are $\frac{9!}{3!}$ possible permutations. Since there are 7 different phrases that can repeat, the number has to be multiplied by 8 resulting in 483840 combinations. Combining this with the number of possible endings, we have $24 \times 483840 = 11612160$ songs for case 1.

   Case 2: 2 different phrases will each repeat once.

   Case 2.1: One of the phrases is "Ronnam Ronnam Nowy Nowy Ronnam Wisda Taichi!", "Eh Eh", or "Chhhhhikuuuuuuuuuun"

   Since the above 3 phrases always has to be at the end of a verse, and there are 4 verses. If one of the repeated phrases is one of the three above, then we can count the number of possible songs by first ordering the last phrase of each verse then the rest of the song.

   Since there are 4 verses and 3 distinct endings where 1 of them repeats. There are $\frac{4!}{2!}$ possible permutations of endings. Since there are 3 different endings that can repeat, the number has to be multiplied by 3 resulting in 36 possible endings.

   For the remaining 8 phrases we have 7 distinct phrases and one of them repeats. Hence we have $\frac{8!}{2!} = 20160$ combinations, since there are 7 different phrases that can repeat, there are $20160 \times 7 = 141120$ combinations for the endings

Combining this with the number of combinations of endings, we have $36 * 141120 = 5080320$ combinations for case 2.1

Case 2.2: Both of the phrases are not “Ronnam Ronnam Nowy Nowy Ronnam Wisda Taichi!”, “Eh Eh”, or “Chhhhhikuuuuuuuuuun”

Again we first order the number of possible endings. Since there are only 3 endings but 4 verses, 1 of the endings will be any phrase. We will leave this blank. Considering the blank to be an element as well we have $4! = 24$ combinations for endings.

For the other 9 phrases, there are 7 distinct phrases where 2 of them repeats once. There are $\frac{9!}{2!2!}$ possible permutations. Since there are $\binom{7}{2=21}$ different combinations phrases that can repeat, the number has to be multiplied by 21 resulting in 1905120 combinations. Combining this with the number of possible endings, we have $24 \times 1905120 = 45722880$ songs for case 2.2

Combining the number of combinations for case 1, 2.1 and 2.2, we have $11612160 + 5080320 + 45722880 = 62415360$

The product of the non-zero digits of 62415360 is 4320, hence the remainder when 4320 is divided by 1000 is $\boxed{320}$

For the other 9 phrases, there are 8 distinct phrases where 1 of them repeats. There are $\frac{9!}{2!}$ possible permutations. Since there are 8 different phrases that can repeat, the number has to be multiplied by 8 resulting in 1451520 combinations. Combining this with the number of possible endings, we have $24 \times 1451520 = 34836480$ songs for case 2.

3. First lets count the number of possible combinations of songs Yan Yau listened to.

   There are 10 possibilities for the first song. Since the mp3 will not play the same song twice in a row, there are 9 possibilities for the second song. Likewise, there are also 9 possibilities for the 3rd, 4th, and 5th songs. In total there are 10×9^4 possible combinations of songs that Yan Yau listened to.

   Now we count the number of combinations of songs where the songs are distinct.

   Since we are choosing 5 songs out of 10 and then ordering them, this is basically $P(10,5) = \frac{10!}{(10-5)!}$

   Hence the probability that all the songs he listened to are distinct is:

$$\frac{\frac{10!}{(10-5)!}}{10 \times 9^4}$$
$$= \frac{10!}{5! \times 10 \times 9^4}$$
$$= \frac{8 \times 7 \times 6}{9^3}$$
$$= \frac{16 \times 7}{3^5}$$
$$= \frac{112}{243}$$

Hence: $\left\lfloor \frac{m+n}{1000} \right\rfloor = \left\lfloor \frac{112+243}{1000} \right\rfloor = \left\lfloor \frac{355}{1000} \right\rfloor = \boxed{0}$

4. First you have to choose 5 places for our special letters (MATHS) i.e. $\binom{26}{5}$, and then others will be arranged in 21! ways so the number of permutations where the permutation contain the letters MATHS in the same order will be $21! \times \binom{26}{5}$

   There are 26! permutations of the letters of the english alphabet, so the probability that the permutation contain the letters MATHS in the same order is: $\frac{21!}{26!} = \frac{21!}{26!} \times \frac{26!}{5!21!} = \frac{1}{5!} = \frac{1}{120}$

   Hence $m + n = 1 + 120 = \boxed{121}$

5. Let base of the nth burger be b_n, meatloaf of nth burger be m_n, bun on top of the nth burger be n_1

   So we have $b_1, b_2, b_3, m_1, m_2, m_3, n_1, n_2, n_3$

   We have a total of $9! = 362880$. However among these arrangements, for b_1 must be done before m_1, m_1 must be done before n_1, There are 3! orderings for b_1, m_1, n_1 but only one where it goes in the order of base-meatloaf-top so we will have to divide by 3! the same goes for b_2, m_2, n_2 and b_3, n_3, m_3 so the total number of ways there are for Tom to produce these 3 burgers is $\frac{9!}{3!3!3!} = 1680$. 1680 divided by 10 is $\boxed{168}$

6. Notice that for Aditya to receive the ball at the 8th pass, Aditya cannot have the ball at the 7th pass

   Therefore, let a_n denote the number of ways for Aditya to have the ball at the nth pass, and b_n to denote the number of ways for either Ben or Cody to have the ball at the nth pass. We notice:

$$a_n = b_{n-1}$$

   and

$$b_n = 2a_{n-1} + b_{n-1}$$

   Therefore we get:

$$b_n = 2a_{n-1} + a_n = a_{n+1}$$

$$\therefore a_n = a_{n-1} + 2a_{n-2}$$

Hence we get a recurrence relation. Working out the relation we get that $a_8 = \boxed{86}$

7. The color of the first card does not matter, as long as you draw a card. After you draw the first card, there are 25 cards left that are the same color as the first card and 26 that are a different color. The probability that the next card is different is equal to $\frac{26}{51}$, so $A = 26$ and $B = 51$. $A + B = \boxed{77}$.

8. There are two cases , either the 3 men and 4 women (with their respective husbands and wives) are in

   (a) the same group

   (b) different groups

   If they are in the same group, we have $\binom{31}{12}$ different groups possible because we have to choose 12 couples out of the 31 remaining to form the group in which the 7 special couples are and the remaining 19 will form other group.

   If they are in different groups then we have $\binom{31}{15}$ groups, as we have fixed the 4 couples of one group, and 3 of the other, hence just need to choose 15 couples out of 31 for the 1st group.

   Hence there are $\binom{31}{12} + \binom{31}{16} = 441660720$ groups and digit sum of the answer is $\boxed{30}$

9. For each student there is a $\frac{19}{20}$ chance that the student does not get his/her own question, and there are 20 students, Hence the probability that no student gets their own question is $\left(\frac{19}{20}\right)^{20} \approx 0.358486$. Hence $\lfloor 1000n \rfloor = \boxed{358}$

10. We will solve this using generating functions. There is 1 way to color no eggs red, 1 way to color 3 eggs red, 1 way to color 6 eggs red, etc. Hence we have a function $1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18}$. Repeating similarly for Green and blue eggs, we have: $1 + x^4 + x^8 + x^{12} + x^{16} + x^{20}$ and $1 + x^5 + x^{10} + x^{15} + x^{20}$.

    The number of ways we can paint the eggs will be the sum of coefficients of $1, x^4, x^8, x^{12}, x^{16}, x^{20}$.

    There is one way to make 1, this is to choose $(1, 1, 1)$ in all 3 brackets.

    There is only 1 way to make x^4, this is to choose $(1, x^4, 1)$.

There are 2 ways to make x^8, this is to choose $(x^3, 1, x^5)$ and $(1, x^8, 1)$

There are 3 ways to make x^{12}, this is to choose (x^3, x^4, x^5), $(x^{12}, 1, 1)$ and $(1, x^{12}, 1)$

There are 4 ways to make x^{16}, this is to choose (x^3,x^8,x^5), $(x^{12}, x^4, 1)$, $(x^6, 1, x^10)$ and $(1, x^{16}, 1)$

There are 6 ways to make x^{20}, this is to choose (x^3,x^{12},x^5), $(x^{12}, x^8, 1)$, (x^6, x^4, x^10), $(x^{15}, 1, x^5)$, $(1, 1, x^{20})$ and $(1, x^{20}, 1)$

Combining the combinations together, we have a total of $1 + 1 + 2 + 3 + 4 + 6 = \boxed{17}$ ways to color the eggs.

Long Questions

1. We know that there are 7 possible remainders $0, 1, 2, 3, 4, 5, 6 \pmod 7$ Consider the case whereby all 7 remainders modulo 7 are present within the 37 numbers. Then, we get: $7|0+1+2+3+4+5+6 = 21$ But what if only 6 (or less) remainders are present? Using the Pigeonhole Principle, we get $37/6 > 6$. Therefore, we get at least 7 numbers of 1 of the 7 remainders present. And we know that 7 times a number will result in a number divisible by 7. Hence given any 37 positive integers it is possible to choose 7 whose sum is divisible by 7

2. First, find dimension of the largest cube that would fit into the unit sphere. Let such a cube have side length x. Then, its main diagonal will be 2 (the diameter of the sphere which the cube lies in). Therefore, $3x^2 = 2^2 = 4$ (by applying Pythagoras twice). Thus, volume of such a cube, or $x^3 = \frac{8}{3\sqrt{3}}$. Then, we know there are $\frac{5\cdot5\cdot5}{x^3}$ such cubes in the large cube. On average, there are $\frac{260x^3}{5\cdot5\cdot5} > 3$ points in each cube. Therefore, there lies at least a small cube with at least 4 points. Hence, it follows that there is a sphere of unit radius that contains at least 4 points.

3. Since n is an odd number and a_n is also odd, there are $\frac{n+1}{2}$ odd numbers and $\frac{n-1}{2}$ even numbers.

   In the given expression when n is an odd number p_n is added to $17^{\frac{n+1}{2}}$ and when n is an even number p_n is added to $6^{\frac{n}{2}}$. 17^n is odd for all $n \in \mathbb{Z}$ and 6^n is even for all $n \in \mathbb{Z}$

   Since n is odd, there will be $\frac{n+1}{2}$ brackets where p_n is added to an odd number and there will be $\frac{n-1}{2}$ brackets where p_n is added to an even number.

   In order for the expression to be not divisible by 2 an odd number needs to be paired with an even number. However there are $\frac{n+1}{2}$ odd p_n but only $\frac{n-1}{2}$ even numbers. Let $\frac{n-1}{2}$ be equal to k, then $\frac{n+1}{2} = k + 1$. There are $k + 1$ odd numbers that have to be paired with k even numbers, by the pigeonhole principle an even number will have to be paired with more than one odd number, which is not possible. Hence the expression must always be divisible by 2.