Competition #5 Solutions

The Junior Online Math Olympiad

26th May 2014 - 3rd June 2014

Answer Key:

1. 255
2. 25
3. 32
4. 9
5. 3
6. 6
7. 401
8. 3
9. 168
10. 986

Short Questions

1. By remainder theorem , when $f(x)$ is divided by $x - a$ then $f(a)$ is the remainder . Thus remainder when $m^5 + 5^5$ is divided by $m - 5$ is $5^5 + 5^5 = 6250$

   As the polynomial $m^5 + 5^5$ is given to be divisible by $m - 5$, we can conclude that remainder should be 0 which means $m - 5 \mid 6250$ and as greatest divisor of 6250 is 6250 itself , we can conclude that for maximum m , $m - 5 = 6250$ hence the last 3 digits of 6255 is $\boxed{255}$

2. By adding 6 to both sides of the first equation, we get: $(x+2)(y+3) = 8$. The only possible solution for x and y is $x = 0$, $y = 1$. Substituting into the second equation we get $z = 12$. Hence our wanted value is $\boxed{25}$

3. For very large values of an integer n, the figure $x^{2n}+y^{2n} \leq k^{2n}$ approaches a square of side length $2k$. When $k=2$, the perimeter of the figure is thus 16 and the area is also 16, hence their sum is $\boxed{32}$

4. Apply AM-HM inequality to a, b, c to get

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Then

$$\frac{a+b+c}{3} \geq \frac{3abc}{ab+bc+ca}$$

Thus

$$\frac{a+b+c}{3} \geq 3 \implies a+b+c \geq \boxed{9}$$

5. For any prime number $p>3$, The only possible remainders modulo 6 are 1 and 5 .For any integer 'n' ,if $n \equiv 2 \pmod 6$ then $2|n$, if $n \equiv 3 \pmod 6$ then $3|n$ and if $n \equiv 4 \pmod 6$ then $2|n$. Thus every prime number $p>3$ has to be either of the form $6k+1$ or of the form $6k-1$. We are given that p is prime. Thus the given other prime p^2+8 can be compared to $(6k \pm 1)^2+8$ if $p>3$ and from the expansion of the bracket , we get $36k^2 \pm 12k+1+8 = 36k^2 \pm 12k+9$ Which gives us $3|p^2+8$ And because p^2+8 is GIVEN to be a prime , we can conclude that $p \leq 3$ and if $p=2$, then $2|p^2+8 \implies p \neq 2 \implies p=3$ Hence the sum of all possible values of p is $\boxed{3}$

6. Because it is given that $7 \nmid xy$, we can conclude that $7 \nmid y$ and also $7 \nmid x$. Hence we can say that **smallest positive** value of k will follow $1 \leq k \leq 6$ and it will be a value from the set $A = 1,2,3,4,5,6$. From given, i.e. $7 \mid 2x+3y \implies 7|6x+9y$. As a fact, $7|7y$ hence $7|(6x+9y)-(7y) \implies 7|6x+2y \implies 7|6x+2y+280y$ ...... (as $7|280y$)... This gives $7|6x+282y$ and as $1 \leq 6 = k \leq 6$ and $6 \in A$, we can say that the smallest positive value of k is $\boxed{6}$

7. Evaluating the last three digits of a number A means find an x such that: $A \equiv x \pmod{1000}$. Because 201 and 1000 are coprime, by Eulers totient theorem we have that: $201^{\varphi(1000)} \equiv 1 \pmod{1000}$. Calculating $\varphi(1000) = 400$, we have that $201^{400} \equiv 1 \pmod{1000}$. It is clear that $201^{402} \equiv 201 \times 201 \equiv \boxed{401} \pmod{1000}$

8. For all the k except 1 and 2 , $f(k)$ is divisible by 2014

   $f(1)=1,\ f(2)=2$

   Hence the remainder is $\boxed{3}$

9. Let $X = \{1,2,3,4,5,6,7\}$ be the set of 7 friends (i.e. Friend 1, 2, 3, 4, 5, 6, and 7).

   Let $Y = \{1,2,3,4,5\}$ be the set of 5 bookshops (i.e. Bookshop 1, 2, 3, 4, and 5).

   Let S be the set of mappings from X to Y. Notice that if we let:

   A_1 be the set of mapping from X to $Y\ \ \{1\}$,

   A_2 be the set of mapping from X to $Y\ \ \{2\}$,

   A_3 be the set of mapping from X to $Y\ \ \{3\}$,

   A_4 be the set of mapping from X to $Y\ \ \{4\}$,

   A_5 be the set of mapping from X to $Y\ \ \{5\}$, Then the set of mappings from X to $Y = \{1,2,3,4,5\}$ (i.e. surjective mapping) is simply $\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}\right|$, where $\bar{A_1}$ refers to the complement of the set A_1, meaning that that it has the mapping from X to $Y = \{1\}$, and so on.

   By the general Principle of Inclusion and Exclusion, we get

   $$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}\right|$$

   $$= |S| - \sum_{i} |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l|$$

   .

   Therefore,

   $$\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}\right| = 5^7 - 5 \cdot 4^7 + \binom{5}{2} \cdot 3^7 - \binom{5}{3} \cdot 2^7 + \binom{5}{4} \cdot 1^7 = 16800 \blacksquare$$

   Hence $\frac{16800}{100} = \boxed{168}$

10. For further simplicity , draw $IF \perp BC$ and $IP \perp AB$

    By pythagoras theorem , $AC = 26$. As BO is median on hypotenuse , $BO = \frac{AC}{2} = 13$. G is centroid so it divides the median in the ratio $2:1$

    $\therefore BG = \frac{13 \times 2}{3} = \frac{26}{3}$ and $GO = \frac{13}{3}$

    As $\triangle ABC$ is right angled at B , it's inradius $IF = IP$ is given by $\frac{AB+BC-AC}{2} = \frac{24+10-26}{2} = 4$ hence $IP = IF = BF = PB = 4 \implies BI = 4\sqrt{2}$

    As E is midpoint of BC , $BE = 12$ and thus $FE = BE - BF = 12 - 4 = 8$ and $IF = 4$ as proved above hence $IE = \sqrt{4^2 + 8^2} = 4\sqrt{5}$

    O and E are midpoints of AC and BC respectively , hence by midpoint theorem on $\triangle ABC, OE = \frac{AB}{2} = 5$

    By the A-A test , $\triangle ABE \sim \triangle GKE$ and $AE : GE = 3 : 1$ thus $GK = \frac{AB}{3} = \frac{10}{3}$ and by the same similarity , $KE = \frac{AE}{3} = 4$

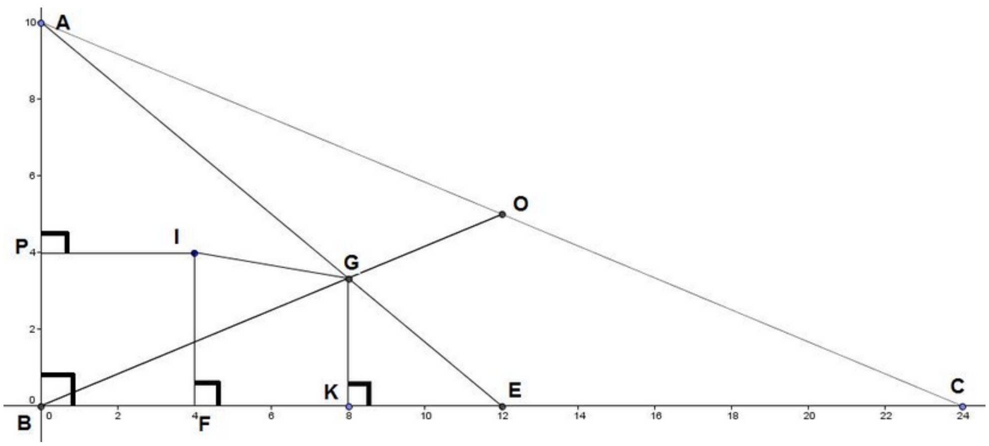
By pyhtagoras theorem on $\triangle GKE$, we get $GE = \sqrt{GK^2 + EK^2} = \sqrt{\frac{100}{9} + 4^2} = \sqrt{\frac{100+144}{9}} = \frac{2\sqrt{61}}{3}$

As $AE = 12, AF = 4, KE = 4$ then $FK = 4$ and as $IF = 4$, $IK = \sqrt{FK^2 + IF^2} = 4\sqrt{2}$

Hence $BG = \frac{26}{3}, GO = \frac{13}{3}, BI = 4\sqrt{2}, IE = 4\sqrt{5}, OE = 5, GE = \frac{4\sqrt{61}}{3}, GK = \frac{10}{3}, KE = 4, IK = 4\sqrt{2}$

Product of these all is $\frac{26}{3} \times \frac{13}{3} \times 4\sqrt{2} \times 4\sqrt{5} \times 5 \times \frac{2\sqrt{61}}{3} \times \frac{10}{3} \times 4 \times 4\sqrt{2}$

$= \dfrac{17305600\sqrt{305}}{81}$ hence $a+b+c = 17305600+305+81 = 17305986$ hence last 3 digits are $\boxed{986}$



Long Questions

1. Since $ACDE$ and $BCFG$ are squares, $AC = CD$ and $BC = CF$.

   We know that angle C is a right angle and that the interior angles of the two squares are also right angles, so therefore the remaining angle $\angle DCF = 90°$.

   By SAS congurence, we have: $\triangle ABC \cong \triangle DFC$.

   Line segment AB in $\triangle ABC$ corresponds to Line segment DF in $\triangle DCF$, therefore $DF = AB$

2. Consider $f(x) = \sin x + \cos x + \tan x$, We know that $f(x)$ is continuous in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ because its composing trigonometric functions also are. Plugging in some values, we see that $f\left(-\frac{\pi}{4}\right) = -1$ and $f(0) = 1$. By Intermediate Value Theorem, we have that there is at least one real number between $-\frac{\pi}{4}$ and 0 where $f(x) = 0$

3. Let $\sin\theta = s$ and $\cos\theta = c$ We know that $s^2 + c^2 = 1$, thus:

$$s^4 + c^4 = 1 - 2s^2c^2 \tag{1}$$
$$s^3 + c^3 = (s+c)(s^2 + c^2 - sc) = (s+c)(1 - sc) \tag{2}$$

$$\therefore (s^4 + c^4)(s^3 + c^3) = (s+c)(1 - 2s^2c^2)(1 - sc)$$
$$\therefore s^7 + c^7 + s^4c^3 + s^3c^4 = (s+c)(1 - sc - 2s^2c^2 + 2s^3c^3)$$
$$\therefore s^7 + c^7 + s^3c^3(s+c) = (s+c)(1 - sc - 2s^2c^2 + 2s^3c^3)$$
$$\therefore s^7 + c^7 = (s+c)(1 - sc - 2s^2c^2 + 2s^3c^3 - s^3c^3)$$
$$\therefore s^7 + c^7 = (s+c)(1 - sc - 2s^2c^2 + s^3c^3)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - sc(1 - s^2c^2)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - \frac{2sc}{2}(1 - s^2c^2)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - \frac{2sc + 1 - 1}{2}(1 - s^2c^2)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - \frac{2sc + s^2 + c^2 - 1}{2}(1 - s^2c^2)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - \frac{(s+c)^2 - 1^2}{2}(1 - s^2c^2)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - \frac{(s+c+1)(s+c-1)}{2}(1 - s^2c^2)$$
$$\therefore s^7 + c^7 = (s+c)(s^4 + c^4 - \frac{(s+c+1)(s+c-1)(1 - s^2c^2)}{2}$$

Hence proved