Competition #6 Solutions

The Junior Online Math Olympiad

30th June 2014 - 7th July 2014

Answer Key:

1. 8
2. 71
3. 611
4. 48
5. 82
6. 0
7. 346
8. 1
9. 0
10. 211

Short Questions

1. Each positive integer can be represented as $2^a + 2^b + 2^c + 2^d + ...$ for some whole numbers a,b,c,d... and some number of such powers of 2 (like 123 can be written as $64 + 32 + 16 + 8 + 2 + 1$ as all are powers of 2 ($1 = 2^0$)

   So the minimum number of boxes exists for boxes having $(1, 2, 4, 8, 16, 32, 64, 16)$ i.e. $\boxed{8}$ boxes

2. $25|n^2(n^3 - 1)$

   Thus if $5|n$ then n satisfies the condition.

Other solutions occur when $25|n^3-1$ Which can be rearranged as $n^3 \equiv 1 \pmod{25}$ And of all the integers till 25, only $n \equiv 1 \pmod{25}$ gives us $n^3 \equiv 1 \pmod{25}$

Hence the numbers which satisfy are : All multiples of 5 less than 300 and numbers of the form $25k+1$ where $0 \le k \le 11$

Hence there are $\boxed{71}$ such n.

3. We see that sum of n terms of this sequence is

$$\sum_{k=1}^{n} k^3 - 4 \times \sum_{k=1}^{n} k - n$$

Hence sum of n terms is also equal to $\left(\frac{n(n+1)}{2}\right)^2 - 4 \times \frac{n(n+1)}{2} - n = \frac{n^4+2n^3-7n^2-12n}{4}$ Hence sum of first 66 terms is 4879611. And the last 3 digits are $\boxed{611}$.

4. Total number of pizzas is $6 \times 7 \times 8 = 336$ He can eat the pizzas having 5 permissible bases,5 flvours and 7 toppings means $5 \times 5 \times 7 = 175$ pizzas. Thus he can't eat $336 - 175 = 161$ pizzas. Hence ratio asked is $\frac{161}{175} = \frac{23}{25}$ and hence answer is $23 + 25 = \boxed{48}$

5. At first , the ship was at a distance of $300\sqrt{3}$ m from the tower as $tan(60°) = \sqrt{3}$(we will take the angle it makes with vertical for calulating the distance). After 40 seconds , the ship was 300 m away from the tower as $tan(45°) = 1$. Thus the speed of the ship is $\frac{300(\sqrt{3}-1)}{40}$ m/s $=$ $\frac{15(\sqrt{3}-1)}{2}$ m/s

The ship needs to travel 300m more to reach the coast and it's speed is constant. Thus it will take $\frac{300 \times 2}{15(\sqrt{3}-1)}$ seconds more i.e. $\frac{40}{\sqrt{3}-1}$ seconds.

Thus $x + y = \frac{15(\sqrt{3}-1)}{2} + \frac{40}{\sqrt{3}-1}$

$= \frac{15(\sqrt{3}-1)}{2} + \frac{40(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \frac{15(\sqrt{3}-1)}{2} + \frac{40(\sqrt{3}+1)}{2} = \frac{55\sqrt{3}+25}{2}$

Hence out answer is $a + b + c = 55 + 25 + 2 = \boxed{82}$

6. Here the numerator is $x^4 - 20x^3 + 150x^2 - 500x + 625 = (x-5)^4$ Hence Equation reduces to $(x-5)^3 = 0$ and has it's only solution as $x = 5$ but at $x = 5$ in the original question , the denominator becomes 0 hence we can't say this is an answer. Thus this equation has $\boxed{0}$ solutions .

7. $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) = 36 \implies 36 = 66+2(ab+bc+ca)$

   Hence we get $ab + bc + ca = -15$

   It is known that $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

   Thus $666 - 3abc = 6 \times (66 + 15) = 486$

   From this we get $abc = 60$

   Now, $(a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2)$

   We have $a^4 + b^4 + c^4 = 66^2 - 2(a^2b^2 + b^2c^2 + c^2a^2)$

   The unwanted $(a^2b^2 + b^2c^2 + c^2a^2)$ thing is found by following method, $(ab + bc + ca)^2 = unwanted + 2(ab^2c + abc^2 + a^2bc)$

   $225 = unwanted + 2(abc(a + b + c))$

   $225 = unwanted + 2 \times 60 \times 6$

   Hence $unwanted = 225 - 720 = -495$

   Thus what we want is $a^4+b^4+c^4 = 66^2+990 = 5346$ and the last 3 digits is $\boxed{346}$

8. If you want a number divisible by 11, " the sum of it's digits on odd places - sum of digits on even places" should be divisible by 11. As we are given that numbers' sum is 13, we can't obtain this difference as 0 (we need even digits sum for making it possible) then, as we want the next number divisible by 11, it's 11 and we can't obtain any other difference in that sum of digits on odd and even places (Because we want that difference of sums divisible by 11 and we are not going to achieve above 13 $\implies$ we have to get this difference as 11. Thus if you want that difference 11, you have to split the set into sums 12 and 1, hence there is the digit $\boxed{1}$ for sure.

9. $x = y^4 + 2y^3 - 21y^2 - 21y - 40$ is $y = x^4 + 2x^3 - 21x^2 - 21x - 40$ reflected through the line $y = x$ Hence the intersections will occur then $y = x$. Thus the sum of all x co-ordinates will be same as sum of all y co-ordinates Hence answer will be $\boxed{0}$.

10. Factor to $(x-y)(x+y)$, and since both are integers and 211 is prime, the only solution is when $x - y = 1$ and $x + y = 211$, so $(x, y) = (106, 105)$. Hence the sum of the values of x and y is equal to $106 + 105 = \boxed{211}$

Long Questions

1. We have that $\frac{x^y+1}{x+1}$ is a natural number if and only if $x = 0$ or y is an odd number. By setting $x = y$, we have that x **must be a odd number** in order for $\frac{x^x+1}{x+1}$ to be a natural number.

2. By AM-HM inequality:

$$\frac{x+y+z}{3} \geq \frac{3xyz}{xy+yz+xz} \implies \frac{x+y+z}{3} \geq \frac{3xyz}{3} = xyz$$

$$\therefore \frac{x+y+z}{xyz} \geq 3 \implies \frac{x+y+z}{x^2y^2z^2} \geq \frac{3}{xyz} \qquad (1)$$

We know that $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$ by identity Hence we get

$$3xyz = x^3+y^3+z^3-(x+y+z)(x^2+y^2+z^2-3)$$

Now , we write -3 as $-1-1-1$ and get $3xyz = x^3+y^3+z^3-(x+y+z)(x^2-1+y^2-1+z^2-1)$ Then factorize them using $a^2-1 = (a+1)(a-1)$ to get the value of xyz as

$$xyz = \frac{x^3+y^3+x^3-(x+y+z)\left[(x+1)(x-1)+(y+1)(y-1)+(z+1)(z-1)\right]}{3}$$

in RHS of equation 1 to get the required result.

3. There are initially 2 odd numbers and 1 even number on the whiteboard. If an odd number is chosen, the sum of the remaining two numbers will be odd. If an even number is chosen, the sum of the remaining two numbers will be even. Hence the sum of the numbers on the whiteboard will always be even. $6^{2012}+7^{2013}+8^{2014} \equiv 1 \pmod 2$. Hence it is impossible to have the numbers: $6^{2012}, 7^{2013}, 8^{2014}$ on the whiteboard at the same time.