Competition #7 Solutions

The Junior Online Math Olympiad

28th July 2014 - 4th August 2014

Answer Key:

1. 22050
2. 123
3. 29
4. 1
5. 2
6. 8
7. 42
8. 5353
9. 284
10. 37

Short Questions

1. In a 20×14 chessboard, there are 21 horizontal lines and 15 vertical lines. To make a rectangle, we have to choose 2 lines out of 21 and then 2 lines out of 15. Hence there are $\binom{21}{2}\binom{15}{2} = \boxed{22050}$ rectangles that can be formed.

2. Expanding, we get: $w^2 - 1 = w$, w is actually the Golden Ratio, and has properties: $w^2 = w + 1$ and $\frac{1}{w} = w - 1$

   Now we find $w^2 + w^{-2}$

$$
\begin{aligned}
w^2 + w^{-2} &= w + 1 + (w-1)^2 \\
&= w + 1 + w^2 - 2w + 1 \\
&= w + 1 + w + 1 - 2w + 1 \\
&= 3
\end{aligned}
$$

Squaring both sides:

$$
\begin{gathered}
w^4 + w^{-4} + 2 = 9 \\
w^4 + w^{-4} = 7
\end{gathered}
$$

Multiplying by $w^2 + w^{-2}$ again:

$$
\begin{gathered}
\left(w^4 + w^{-4}\right)\left(w^2 + w^{-2}\right) = 7 \times 3 \\
w^6 + w^2 + w^{-2} + w^{-6} = 21 \\
w^6 + 3 + w^{-6} = 21 \\
w^6 + w^{-6} = 18
\end{gathered}
$$

Multiplying by $w^4 + w^{-4}$:

$$
\begin{gathered}
\left(w^6 + w^{-6}\right)\left(w^4 + w^{-4}\right) = 18 \times 7 \\
w^{10} + w^2 + w^{-2} + w^{-10} = 126 \\
w^{10} + w^{-10} = \boxed{123}
\end{gathered}
$$

3. Let a_n be the number of regions in which n lines divide a plane. Then we see $a_0 = 1$ (0 lines will keep the plane as a whole region), and when you draw the n^{th} line, it will interect maximum all of the $n-1$ lines that were drawn before, adding n regions in the number a_{n-1}. Thus we conclude that $a_n = a_{n-1} + n$. Using this recurrence relation, we write each term in terms of it's previous term repeatedly to finally get

$$a_n = a_0 + 1 + 2 + 3 + ... + (n-1) + n$$

$$a_n = a_0 + \frac{n(n+1)}{2} = 1 + \frac{n(n+1)}{2}$$

At $n = 7$, we get the number to be $\boxed{29}$.

4. For all numbers in S, if the number ends in 2 or 0 (except 2) then it will be divisible by 2, and hence it is not prime.

If the number in S ends in 1 or 3, the number will be divisible by 3.

Hence there is only $\boxed{1}$ number in S that is prime, namely 2.

5. By Cauchy-Schwartz,

$$(1+1)(x^2+y^2) \geq (x+y)^2,$$

so $2 \geq \dfrac{(x+y)^2}{x^2+y^2}$.

Since equality is achievable (when $x = y = 1$, for example), $M = \boxed{2}$.

6. If Yan Yau arrives at anytime between 10:00-10:10, 10:16-10:22, 10:28-10:40, 10:46-10:52, 10:58-11:00, he will take the C9 bus. Hence there is a $\frac{36}{60} = \frac{3}{5}$ chance he takes the C9. Hence $a + b = 3 + 5 = \boxed{8}$

7. If Yan Yau arrives at anytime between 10:00-10:04, he will take the C9 bus at 10:04. Hence there is a $\frac{4}{60}$ chance he takes the C9 at 10:04, arriving 8 minutes after 10:00

   If Yan Yau arrives at anytime between 10:04-10:10, he will take the C9 bus at 10:10. Hence there is a $\frac{6}{60}$ chance he takes the C9 at 10:10, arriving 14 minutes after 10:00

   If Yan Yau arrives at anytime between 10:10-10:16, he will take the C4 bus at 10:16. Hence there is a $\frac{6}{60}$ chance he takes the C4 at 10:16, arriving 25 minutes after 10:00

   If Yan Yau arrives at anytime between 10:16-10:22, he will take the C9 bus at 10:22. Hence there is a $\frac{6}{60}$ chance he takes the C9 at 10:22, arriving 26 minutes after 10:00

   If Yan Yau arrives at anytime between 10:22-10:28, he will take the C4 bus at 10:28. Hence there is a $\frac{6}{60}$ chance he takes the C4 at 10:28, arriving 37 minutes after 10:00

   If Yan Yau arrives at anytime between 10:28-10:34, he will take the C9 bus at 10:34. Hence there is a $\frac{6}{60}$ chance he takes the C9 at 10:34, arriving 38 minutes after 10:00

   If Yan Yau arrives at anytime between 10:34-10:40, he will take the C9 bus at 10:40. Hence there is a $\frac{6}{60}$ chance he takes the C9 at 10:40, arriving 44 minutes after 10:00

   If Yan Yau arrives at anytime between 10:40-10:46, he will take the C44) bus at 10:46. Hence there is a $\frac{6}{60}$ chance he takes the C4 at 10:46, arriving 55 minutes after 10:00

   If Yan Yau arrives at anytime between 10:46-10:52, he will take the C9 bus at 10:52. Hence there is a $\frac{6}{60}$ chance he takes the C9 at 10:52, arriving 56 minutes after 10:00

   If Yan Yau arrives at anytime between 10:52-10:58, he will take the C4 bus at 10:58. Hence there is a $\frac{6}{60}$ chance he takes the C4 at 10:58, arriving 67 minutes after 10:00

If Yan Yau arrives at anytime between 10:58-11:00, he will take the C9 bus at 11:04. Hence there is a $\frac{2}{60}$ chance he takes the C9 at 11:04, arriving 68 minutes after 10:00

Let the number of minutes after 10:00 of which Yan Yau arrives be X, then:

$E(X) = \frac{4}{60} \times 8 + \frac{6}{60} \times 14 + \frac{6}{60} \times 25 + \frac{6}{60} \times 26 + \frac{6}{60} \times 37 + \frac{6}{60} \times 38 + \frac{6}{60} \times 44 + \frac{6}{60} \times 55 + \frac{6}{60} \times 56 + \frac{6}{60} \times 67 + \frac{2}{60} \times 68 = 42.1$

Hence when rounded, the expected number of minutes is $\boxed{42}$

8. By the converse of theorem on equal ratios, we conclude that

$$\frac{a_i}{b_i} = k \qquad \forall i \in (1, 2, 3, 4)$$

Hence we want to find sequences of length 4 of $< a_i >$ and $< b_i >$ such that $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4}$ and are integers in the range [1,20]. We know that only numbers which have at least 4 multiples in the range [1,20] are $1, 2, 3, 4$ and 5. Hence we will have 19 different fractions possible (fractions are the values of k possible), namely $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{3}, \frac{2}{5}, \frac{3}{4}, \frac{3}{5}, \frac{4}{5}$ and their reciprocals. (only 1 won't give a different reciprocal and 2/4 is same as 1/2). Thus the equal ratios $\frac{a_i}{b_i}$ can have 19 distinct values.

We'll find the number of sequences a_i only, because accordingly, the b_is will take their values.And that will always be $\binom{q}{4}$ where q is defined as number of multiples less than 21, of the number maximum out of x and y where x and y are numbers included to make the fraction.

See that 5 will be the maximum number in 8 fractions, namely $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ and their reciprocals.

See that 4 will be the maximum number in 4 fractions, namely $\frac{1}{4}, \frac{3}{4}$ and their reciprocals.

See that 3 will be the maximum number in 4 fractions, namely $\frac{1}{3}, \frac{2}{3}$ and their reciprocals.

See that 2 will be the maximum number in 2 fractions, namely $\frac{1}{2}, \frac{2}{1}$.

See that 1 will be the maximum number in 1 fraction, namely $\frac{1}{1}$.

So according to the above said reasoning, the number fractions in this will be $8 \times \binom{4}{4} + 4 \times \binom{5}{4} + 4 \times \binom{6}{4} + 2 \times \binom{10}{4} + \binom{20}{4} = \boxed{5353}$

Hence the answer is:

$$\binom{20}{4}+2\times\left(\binom{10}{4}+\binom{6}{4}+\binom{5}{4}+\binom{4}{4}+\binom{6}{4}+\binom{4}{4}+\binom{5}{4}+\binom{4}{4}+\binom{4}{4}\right)=\boxed{5353}$$

9. We will use the angle bisector property in $\triangle ABC$,

   See that $\dfrac{DC}{BD}=\dfrac{b}{c}\implies\dfrac{BD}{BC}=\dfrac{c}{b+c}$

   Thus we have $BD=\dfrac{ac}{b+c}$

   Now, we use angle bisector property for $\triangle BAD$,

   $\dfrac{BD}{AB}=\dfrac{ID}{AI}=\dfrac{\frac{ac}{b+c}}{c}=\dfrac{a}{b+c}$

   Thus we get $\dfrac{ID}{AD}=\dfrac{a}{a+b+c}$

   Now, $AD=\dfrac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2-a^2}$ (length of the angle bisector)

   So we get the value of $ID=\dfrac{a}{a+b+c}\times\dfrac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2-a^2}$

   Similarly, we get $IF=\dfrac{c}{a+b+c}\times\dfrac{\sqrt{ab}}{a+b}\sqrt{(b+a)^2-c^2}$

   Then , after taking their ratio, answer comes as $\dfrac{ID}{IF}=\dfrac{(b+a)\sqrt{[(b+c)^2-a^2]a}}{(b+c)\sqrt{[(a+b)^2-c^2]c}}$

   And, finally, putting values, $\dfrac{ID}{IF}=\dfrac{b+14}{b+20}\sqrt{\dfrac{(b+6)14}{(b-6)20}}$

   And when you put $b=\dfrac{438}{53}$, answer is $\dfrac{177}{107}$, hence the asked value is $\boxed{284}$.

10. There are 3 ways $x^{x^5-37x^4+7x^3+84x^2+2x+9}=1$ is satisfied.

    (a)
    $$x=1$$

    (b)
    $$x^5-37x^4+7x^3+84x^2+2x+9=0$$

    (c)
    $$x=-1; x^5-37x^4+7x^3+84x^2+2x+9\equiv 0 \pmod 2 \text{ when } x=-1$$

    For case (c), substituting $x=-1$ into $x^5-37x^4+7x^3+84x^2+2x+9$, the expression evaluates to $46\equiv 0 \pmod 2$. Hence $x=-1$ is a solution.

    For case (b), using Vieta's formula, the sum of the roots of $x^5-37x^4+7x^3+84x^2+2x+9$ is equal to 37.

    Hence the sum of the solutions for x is $1+(-1)+37=\boxed{37}$

Long Questions

1. We know that k must be a power of two, so let $k = 2^n$ where $n \in \mathbb{Z}$. So we have: $2^{2^n} = 2^{2n}$.

   Taking the log of both sides we have $2^n = 2n$. The only possible values for n is $n = 1, 2$, any value of $n > 2$ will result in $2^n > 2n$.

   Hence the only possible integer values of k is $k = 2, 4$

2. $3 \uparrow\uparrow 2014$ expressed as a exponenetial tower is: $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}$ where there are 2014 threes.

   The last digit of exponents of threes repeat with the cycle: $1, 3, 9, 7$ where the last digit equals 1 when the power 3 is raised to can be expressed as $4n$ for some positive integer n. The last digit will be 3 when the power 3 is raised to can be expressed as $4n + 1$ for some positive integer n, and so on.

   We have to find $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}} \pmod 4$ where there are 2013 threes.

   $$3 \equiv -1 \pmod 4$$

   When k is an odd number:

   $$3^k \equiv -1^k \equiv -1 \pmod 4$$

   $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}$ will always be an odd number no matter how many threes there are, hence $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}} \equiv -1 \pmod 4$. This means $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}} = 4n + 3$ for some n. Hence the last digit is $\boxed{7}$

3. Rewriting the equation as:

   $$x^4 - 10x^2 + 25 = 12x^2 + 12x + 3$$

   We can factorise both sides into:

   $$(x^2 - 5)^2 = 3(2x + 1)^2$$

   Thus we have:

   $$x^2 - 5 = \pm\sqrt{3}(2x + 1)$$

   Solving these two equations:

   $$x^2 - 5 = \sqrt{3}(2x + 1)$$
   $$x^2 - 2\sqrt{3}x - \sqrt{3} - 5 = 0$$

$$x = \frac{2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4(1)(-\sqrt{3}-5)}}{2}$$
$$x = \sqrt{3} \pm \sqrt{8+\sqrt{3}}$$

Solving the second equation:

$$x^2 - 5 = -\sqrt{3}(2x+1)$$
$$x^2 + \sqrt{3}2x + \sqrt{3} - 5 = 0$$
$$x = \frac{-2\sqrt{3} \pm \sqrt{(-2\sqrt{3})^2 - 4(1)(\sqrt{3}-5)}}{2}$$
$$x = -\sqrt{3} \pm \sqrt{8-\sqrt{3}}$$

Since the original equation is a 4th degree equation, there are only a maximum of 4 distinct roots for x, hence the 4 roots of $x^4 - 22x^2 - 12x + 22 = 0$ are: $x_1 = \sqrt{3} + \sqrt{8+\sqrt{3}}$, $x_2 = \sqrt{3} - \sqrt{8+\sqrt{3}}$, $x_3 = -\sqrt{3} + \sqrt{8-\sqrt{3}}$, and $x_4 = -\sqrt{3} - \sqrt{8-\sqrt{3}}$.