Competition #7 Solutions

The Junior Online Math Olympiad

28th July 2014 - 4th August 2014

Answer Key:

1. 64
2. 23
3. 50
4. 1280
5. 19
6. 112
7. 363
8. 12
9. 77
10. 11

Short Questions

1. Because $1859 = 11 \cdot 13^2 \rightarrow \varphi(1859) = 1560$ and $1573 = 13 \cdot 11^2 \rightarrow \varphi(1573) = 1320$, we are left to evaluate $\varphi(240)$. Seeing $240 = 2^4 \cdot 3 \cdot 5$, we have the answer is $\boxed{64}$

2. See that the asked value is, when expanded, $\dfrac{abcd + bcde + cdea + deab + eabc}{abcde}$.

   By Vieta's formula for co-efficient, in quintic polynomial, the co-efficient of x is the sum of roots taken 4 at a time, which is numerator. And product of roots is the -ve of the constant term,which is denominator. Hence our desired sum is $\dfrac{-92}{-4} = \boxed{23}$

3. Consider the two sets, $\{(a+b),(b+c),(c+d),(d+e)\}$ and $\{\frac{1}{a+b},\frac{2}{b+c},\frac{3}{c+d},\frac{4}{d+e}\}$
.

For two sets of a_i and b_i , we have cauchy schwartz inequality as

$$\left|\sum a_i b_i\right|^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

Applying this to our chosen sets, we get

$$(1+2+3+4)^2 \leq ((a+b)^2+(b+c)^2+(c+d)^2+(d+e)^2)\left(\frac{1}{(a+b)^2}+\frac{4}{(b+c)^2}+\frac{9}{(c+d)^2}+\frac{16}{(d+e)^2}\right)$$

After expanding, we see that

$$(1+2+3+4)^2 \leq 2(\frac{a^2}{2}+\frac{e^2}{2}+b^2+c^2+d^2+ab+bc+cd+de)\left(\frac{1}{(a+b)^2}+\frac{4}{(b+c)^2}+\frac{9}{(c+d)^2}+\frac{16}{(d+e)^2}\right)$$

When you expand the second bracket by taking the two terms at once, (add the 1st two fractions, and then add the later two fractions), we get what is in the needed term.

And hence,

$$50 \leq \textbf{needed term}$$

which gives answer as $\boxed{50}$

4. Since it has to be symmetrical, we only have to consider the first 5 strips. There are 5 possibilities for the first strip, 4 for the second, 4 for the third, 4 for the forth, and 4 for the last strip. So there are $5\times4\times4\times4\times4 = \boxed{1280}$ Tshirts.

5. See that when we divide by $x+2$, by remainder theorem, we just put the value $x=-2$. Thus, we have $p((-2)^6) = p(64) = \pi$.

   Thus, whenever we have $x^6 = 64$, the remainder doesn't change.

$$x^6 - 64 = (x^3-8)(x^3+8) = (x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$$

Sam's other polynomial is $x^3-4x^2+8x-8 = (x-2)(x^2-2x+4)$. Because this is a factor of x^6-64, this will also give the same remainder, hence we conclude $R=\pi$.

$$\prod_{k=0}^{3} \cos\left(\frac{(2k+1)R}{18}\right) = \cos(\frac{\pi}{18})\cos(\frac{3\pi}{18})\cos(\frac{5\pi}{18})\cos(\frac{7\pi}{18})$$

And this is same as

$$\cos(10^\circ)\cos(30^\circ)\cos(50^\circ)\cos(70^\circ) = \cos(10^\circ)\frac{\sqrt{3}}{2}\cos(60^\circ-10^\circ)\cos(60^\circ+10^\circ) = \frac{\sqrt{3}}{2}\frac{\cos(3\times10^\circ)}{4} = \frac{3}{16}$$

And hence the asked value is $3+16 = \boxed{19}$

6. Say each of them ate k cookies at the end.

   Thus, there were $3k+1$ cookies when they came home. And this was $\frac{2}{3}$ of what was remaining after Yan Yau ate at his last visit. 1 was given to dog, so when Cody had gone, there were $(3k+1)\frac{3}{2}+1$ cookies remaining, similarly doing the same process of thinking, we get that when Aditya had come home for the first time, the number of cookies was

$$\left(\left(((3k+1)\frac{3}{2}+1)\frac{3}{2}+1\right)\frac{3}{2}+1\right)$$

   This is, after expanding,

$$\frac{81k+65}{8}$$

   . Because no cookie was broken in the process, we have $\dfrac{81k+65}{8} \in \mathbb{N}$.

   Simply, $\dfrac{81k+65}{8} = \dfrac{80k+64+k+1}{8} \in \mathbb{N} \implies \dfrac{k+1}{8} \in \mathbb{N}$, and the minimum positive integer k here will be 7.

   If everyone ate 7 cookies at the end, we work backwards and get that there were 79 cookies made in morning.

   Thus, in Aditya's first visit,he gave 1 cookie to dog and ate 26 cookies.

   But there is something more too ! At night, Aditya ate 7 cookies with everyone, so he, in total ate $26+7=33$ cookies.

   The asked answer is $79+33=\boxed{112}$.

7. By GP formula $a_{n-k}\cdot a_{n+k}={a_n}^2$, we have that the second term is 9^2 and the fourth is 3^2. Thus the third term is 27, the first term is 1 and the fifth one is 243. Their sum is $\boxed{363}$

8. Boringly elevating the first equation to the fifth power, to the third power, and substituting using Vieta's, we find out the roots are:
   $(2,-1);(-1,2);(\frac{1+\sqrt{11}i}{2},\frac{1-\sqrt{11}i}{2});(\frac{1-\sqrt{11}i}{2},\frac{1+\sqrt{11}i}{2})$.
   Hence the sum is:

$$2+1+1+2+2\sqrt{\left(\frac{1}{2}\right)+\left(\frac{\sqrt{11}}{2}\right)^{2^{2}}}=2+1+1+2+2(3)=\boxed{12}$$

9. Equality occurs only when $2x^2-x=9x+12$ or when $(2x^2-x)+(9x+12)=36$

   Considering the first case

$$2x^2-10x-12=0$$
$$x^2-5x-6=0$$

$$(x-6)(x+1)=0$$

Hence $x = 6$ or -1.

Considering the second case:

$$2x^2 + 8x - 24 = 0$$
$$x^2 + 4x - 12 = 0$$
$$(x+6)(x-2) = 0$$

Hence $x = -6$ or 2.

So the sum of the squares of integers that satisfy the equation is: $6^2 + (-1)^2 + (-6)^2 + 2^2 = \boxed{77}$

10. Note that $(a-1)^4 \geq 0$ is equivalent to $\frac{a^3+1}{a^2+1} \geq \sqrt[4]{\frac{a^4+1}{2}}$

    So, summing up similar inequalities, and noting that $\frac{a^3+1}{a^2+1} = a - \frac{a-1}{a^2+1}$, we get $4 - \sum_{cyc} \frac{a-1}{a^2+1} \geq \sum_{cyc} \sqrt[4]{\frac{a^4+1}{2}}$. Finally, use AM-GM on the RHS then rearrange to get $C \leq 2^{\frac{7}{4}}$. Equality holds iff $a = b = c = d = 1$, so $C \geq 2^{\frac{7}{4}}$. This gives $C = 2^{\frac{7}{4}}$ and $x + y = \boxed{11}$.

Long Questions

1. Assuming the contrary, let $1-k$ be rational. This means that $1 - k = \frac{m}{n}$ for some integers m and n.

   $k = -1((1-k)-1) = -1((\frac{m}{n})-1) = 1 - \frac{m}{n} = \frac{n-m}{n}$

   Since n and m are both integers $n-m$ is also an integer and hence $\frac{n-m}{n}$ is a rational number. This contradicts the given condition that k is irrational. Hence $1-k$ is also irrational

2. This question has been removed and omitted from marking, the word "non-trapezoid" was unnecessary in this question.

3.
$$19 \mid 19x + 19y + 19z \quad (1)$$
$$19 \mid 7x + 5y - 3z \quad (2)$$

   $(1) - 2 \times (2)$:

$$19 \mid 5x + 9y + 25z$$
$$19 \mid 5x + 9y + 25z - 38z$$
$$\boxed{19 \mid 5x + 9y - 13z}$$

   Hence Proved