Competition #9 Solutions

The Junior Online Math Olympiad

22nd September 2014 - 29th September 2014

Answer Key:

1. 198
2. 172
3. 2554
4. 1470
5. 17
6. 0
7. 4
8. 149
9. 1
10. 46

Short Questions

1. All 5 digit permutations of 1, 3, 5, 7 and 9 will be larger than 3900, so there are $5! = 120$ five digit numbers Guilherme can make.

   When the first two digits are 39, there are $P(3,2) = 6$ choices for the last two digits.

   For four digit numbers starting with 5, 7, or 9. There are 3 possibilities for the first digit, 4 possibilities for the second, 3 for the third, and 2 for the last digit. Hence there are $3 \times 4 \times 3 \times 2 = 72$ four digit numbers starting with 5, 7, or 9.

   Any other combinations of thsoe digits will result in a number less than 3900, hence the total number of possible numbers Guilherme can make is $120 + 6 + 72 = \boxed{198}$

2. We see that $2014 \equiv 8 \equiv 2^3 \pmod{17}$

   See that $2^4 \equiv -1 \pmod{17}$

   Using Fermat's Little Theorem , $2014^{16} \equiv 8^{16} \equiv 1 \pmod{17}$. As we have $2014 \equiv 14 \pmod{16}$, simply $8^{2014} \equiv 8^{14} \equiv 2^{42} \equiv 2^{4\times 10} \times 2^2 \equiv (-1)^{10} times 4 \equiv 4 \pmod{17}$ Hence $O = 4$.

   We get $2014^{2014} \equiv 2 \pmod{7}$. Next, using Chinese remainder theorem, $2014^{2014} \equiv 2 \pmod{14}$. Hence $J = 2$.

   We have $2014 \equiv -1 \pmod{31}$ so for M, easily we have $2014^{2014} \equiv (-1)^{2014} \equiv 1 \pmod{31}$. Hence $M = 1$.

   $\overline{JOMO} = 2414 \equiv 400 \pmod{2014}$, and $\varphi(2014) = 936 \implies 2414^{2414} \equiv 400^{542} \pmod{2014}$.

   Order of 400 modulo 2014 is 27, so $400^{27} \equiv 400 \pmod{2014}$.

   This reduces our calculations to $400^{542} \equiv 400^{27\times 20+2} \equiv 400^{22} \pmod{2014}$.

   Now calculating manually, $400^{11} \equiv 780 \pmod{2014} \implies ans \equiv 780^2 \equiv \boxed{172} \pmod{2014}$.

3. The first 10 Geometric means will be $G_1, G_2, ..., G_{10} = 2^{11}, 2^{22}, ..., 2^{110}$

   Then, the further geometric means g_i will be $g_1, g_2, ..., g_{10} = 2^1, 2^2, ..., 2^{10}$, then exclude 2^{11} as it is G_1, making the next thing $g_{11}, g_{12}, ..., g_{20} = 2^{12}, 2^{13}, ..., 2^{21}$, and similarly you get all of the geometric means.

   J=2, O=4, M=1.

   $G_2 = 2^{22}, G_4 = 2^{44}, G_1 = 2^{11}$, gives $\log_2 G_2 + \log_2 G_4 + \log_2 G_1 + \log_2 G_4 = 22 + 44 + 11 + 22 = 99$.

   $g_{\overline{JO}} = g_{24} = 2^{26}, g_{\overline{MO}} = g_{14} = 2^{15}$, gives $\log_2 g_{\overline{JO}} + \log_2 g_{\overline{MO}} = 26 + 15 = 41$

   Asked value is $99 + 41 + 2414 = \boxed{2554}$.

4. Every digit in the base 4 representation of n is a 0, 1, 2, or 3. Since $144 = 2^4 \cdot 3^2$, there must be exactly four 2s and two 3s in the representation. We then have three cases on what the other two digits are:

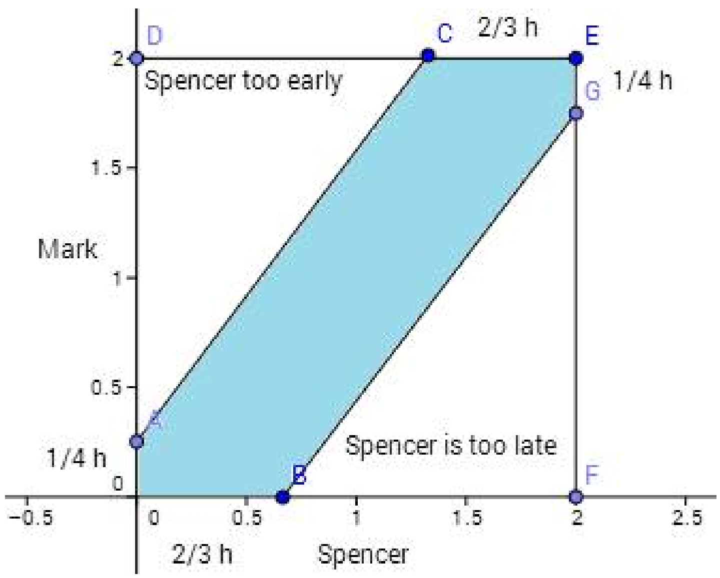
   If the two digits are both 0s, then since a number cannot begin with 0, there are $\binom{7}{2} = 21$ ways to place the zeroes; then, there are $\binom{6}{2} = 15$ ways to place the threes, and the twos are placed in the remaining places, so we have $21 \cdot 15 = 315$ numbers in this case.

   If the two digits are a 0 and a 1, then there are 7 places to put the zero. After that, we are arranging the digits $1, 2, 2, 2, 2, 3, 3$, so there are $\frac{7!}{1!4!2!} = 105$ ways to do so. Thus, we get $7 \cdot 105 = 735$ numbers in this case.

Finally, if the two digits are both 1, then any arrangement of the digits will work (since there are no zeroes). Therefore, we wish to arrange $1, 1, 2, 2, 2, 2, 3, 3$, so there are $\frac{8!}{2!4!2!} = 420$ numbers in this case.

Adding the cases, we have $315 + 735 + 420 = \boxed{1470}$ possible values of n

5. Note that we can represent the problem into the geometric diagram as seen below. Note there are 2 hours between 5pm to 7pm and Mark waits 15 minutes or 1/4 h; similarly Spencer waits for 40 minutes or 2/3 h before leaving. WLOG, let the y-axis represent the timeline for Mark and the x-axis represent that for Spencer. Then, the points in the shaded portion of the graph as shown are the possible occurrences that Spencer and Mark will meet. Thus, to find area of this shaded region over the total area of the 2×2 square is the probability they will meet. Hence, probability is $1 - \frac{\frac{4}{3} \times \frac{7}{4}}{4} = \frac{5}{12}$. Hence $a + b = 5 + 12 = \boxed{17}$



6. Let n be the degree of P. Then, $2n = n^2$, so $n = 0$ or 2.

If $n = 0$, then $P(x) = C$ for some constant C. Since $2C = C$, $C = 0$, so $P(x) = 0$. It is obvious that this polyomial satisfies the condition.

If $n = 2$, then $P(x) = ax^2 + bx + c$ for some real a, b, c. Comparing the coefficients of x^4 of both sides, we have $a = a^3$, which gives $a = 1$, since $a = 0$ implies $P(x)$ is linear, a contradiction. Also, comparing the coefficients of x^3 gives $b = 0$. Finally, comparing the constant term gives $2c = c^2 + c$, which gives $c = 0$ or 1. However, it is easy to see that both $P(x) = x^2$ and $P(x) = x^2 + 1$ doesn't satisfy the equation, thus the only possible value of $P(1)$ is $\boxed{0}$.

7. $\lceil x \rceil - \lfloor x \rfloor$ evaluates to 1 when x is not an integer and 0 when x is an integer.

   The positive values of x where $\tan x = 0$ is $\{\pi, 3\pi \cdots\}$, There are no integers in this set, and hence $\lceil x \rceil - \lfloor x \rfloor$ will evaluate to 1.

   The positive values of x where $\tan x = 1$ is $\{\frac{\pi}{4}, \frac{5\pi}{4} \cdots\}$. All of these are not integers, and hence $\lceil x \rceil - \lfloor x \rfloor$ will evaluate to 1, an equality. Hence the smallest positive value that satisfies the equation is $\frac{\pi}{4}$. Thus $\frac{\pi}{R} = \boxed{4}$

8. Let the number of ways in which Adi can climb up a staircase of n stairs be a_n. Because he can take 1 or 2 or 3 steps at a time, he could directly go to the n^{th} step from either the $(n-1)^{th}$ or $(n-2)^{th}$ or from the $(n-3)^{th}$ step. This gives the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Calculating manually, $a_1 = 1$ (obviously one way to climb 1 stair), $a_2 = 2$ (either go by two 1s or by one 2) , $a_3 = 4$ (either one 3, or three 1s or one 1 followed by one 2, or one 2 followed by one 1). From this , calculating a_9, it's $\boxed{149}$.

9. Note that $\varphi(2015) = 1440$. So, by Euler's Theorem, $7201^{1440} \equiv 1 \pmod{2015}$. Now, since $7200 = 1440 \cdot 5$, the power of 7201 is divisible by 1440. So, the remainder is $\boxed{1}$.

10. Drawing the distance-time graph gives that the point of intersection of Melody and Cody's path is the centroid of the triangle formed by the "vertices" denoting Cody's house, Melody's house and Zi Song's house. Thus, by the centroid theorem, they meet at 6:30pm + $\frac{2}{3}$(15 minutes) = 6:40pm. Hence $x + y = \boxed{46}$

Long Questions

1. Let the consecutive sum be $S(x) = (x-2)^5 + (x-1)^5 + x^5 + (x+1)^5 + (x+2)^5 = 5(x^5 + 20x^3 + 34x)$. Notice it satisfies the given conditions, and that clearly it already is always divisible by 5. We now are left to prove that $S'(x) = x^5 + 20x^3 + 34x$ is always divisible by 5. Taking this equation modulo 5, it becomes $S''(x) = x^5 + 0x^3 + (-1)x = x(x^4 - 1)$. Because all integers modulo 5 can be expressed in the forms $5k-2, 5k-1, 5k, 5k+1 or 5k+2$, it suffices to substitute $x = (-2, -1, 0, 1, 2)$ into $S''(x)$. We have that $-S''(-2) = S''(2) = 30, -S''(-1) = S''(1) = 0 and S''(0) = 0$, so we have finished the proof that the sum of the fifth powers of five consecutive integers is always divisible by 25.

2. Initially substituting $x = 2$ or $x = 3$ satisfies the equation, so 2 and 3 are roots. Expanding the equation, we have $2x^4 - 20x^3 + 78x^2 - 140x + 96 = 0$. By synthetic division, we reach the factoring $2(x-2)(x-3)(x^2-5x+8) = 0$. Solving now $x^2 - 5x + 8 = 0$, we have $x = \frac{5+\sqrt{7}i}{2}$ or $x = \frac{5-\sqrt{7}i}{2}$.

3. Note that $4n^3 + 6n^2 + 4n + 1 = (n+1)^4 - n^4$. If this is equal to k^4 for some positive integer k, then it contradicts Fermat's Last Theorem. Thus, this is never possible.